

Question 7

A third-order equation has an exponentially fading memory:

$$(D + 1)^3 y(t) = \int_0^t e^{-(t-v)} y(v) dv + e^{-t}, \quad (y, y', y'')(0) = (0, 0, 0).$$

All functions are defined for $t \geq 0$.

Tasks

1. Transform the convolution equation and solve algebraically for $Y(s)$. Factor the resulting fourth-degree denominator.
2. Invert the transform in a real form, displaying a useful decomposition in $z = s + 1$.
3. Derive an equivalent ordinary fourth-order initial-value problem. Explain why differentiation requires an additional initial condition, and prove that your fourth-order solution really satisfies the original memory equation.
4. Determine the long-time limit and justify any final-value theorem used. Explain how the feedback changes the zero-state response from what it would be if the memory integral were omitted.

Question 7 – Solution

Strategy. Convert memory into a rational factor, then guard against the extra solutions that differentiation can introduce.

Step 1: Transform and factor. The convolution theorem gives

$$(s+1)^3 Y = \frac{Y}{s+1} + \frac{1}{s+1}, \quad \boxed{Y = \frac{1}{(s+1)^4 - 1}}.$$

The denominator is $s(s+2)((s+1)^2+1)$, with roots $0, -2, -1 \pm i$. A sufficient convergence half-plane is $\operatorname{Re} s > 0$.

Step 2: Invert real quadratic factors. For $z = s+1$,

$$\frac{1}{z^4 - 1} = \frac{1}{2} \left(\frac{1}{z^2 - 1} - \frac{1}{z^2 + 1} \right), \quad \boxed{y(t) = \frac{1}{2} e^{-t} (\sinh t - \sin t)}.$$

Equivalently, $y = (1 - e^{-2t})/4 - e^{-t} \sin t/2$.

Step 3: Prove equivalence, including the lost condition. If $m(t) = \int_0^t e^{-(t-v)} y(v) dv$, then $(D+1)m = y$ and $m(0) = 0$. Applying $D+1$ to the original equation gives

$$((D+1)^4 - 1)y = 0, \quad (y, y', y'', y''')(0) = (0, 0, 0, 1).$$

The fourth value follows from the original equation at zero. The displayed solution has $y = t^3/6 + O(t^4)$ and lies in the four characteristic modes, so it satisfies this fourth-order IVP. Conversely its residual $R = (D+1)^3 y - m - e^{-t}$ obeys $R' + R = 0$ and $R(0) = 0$. Hence $R \equiv 0$, proving equivalence rather than just necessity.

Step 4: Interpret the surviving mode. The explicit formula gives $\boxed{\lim_{t \rightarrow \infty} y(t) = 1/4}$. This also equals $\lim_{s \rightarrow 0} sY$; after cancellation, the poles of sY are $-2, -1 \pm i$, all strictly stable. Without memory, the transform is $1/(s+1)^4$ and the response is $t^3 e^{-t}/6 \rightarrow 0$. Feedback introduces a zero characteristic root and thus a nonzero constant component, although the external forcing itself decays.