

Question 6

A variable coefficient makes the transform satisfy a differential equation. Seek a real-analytic solution at $t = 0$ of

$$ty''' + 3y'' + ty' + y = 0, \quad y(0) = 1, \quad y'(0) = 0,$$

for $t \geq 0$, with a Laplace transform. A third initial value is not specified.

Tasks

1. Determine the only compatible value of $y''(0)$. Explain why the usual three-free-data theorem for a regular third-order equation does not apply at the origin.
2. Use $\mathcal{L}\{tf(t)\} = -F'(s)$ to derive and solve an equation for $Y(s)$ on real $s > 0$. Fix the integration constant from the behavior as $s \rightarrow \infty$.
3. Invert the result by writing it as an integral of elementary cosine transforms. Extend the answer to $t = 0$ and verify the original differential equation.
4. Derive the power-series coefficient recurrence and prove uniqueness among analytic solutions with the prescribed data. Check the first three terms of the large- s expansion of Y against the initial derivatives.

Question 6 – Solution

Strategy. Use the singular-endpoint constraint before transforming; the transform's boundary condition replaces its integration constant.

Step 1: Enforce compatibility. At $t = 0$, continuity of the equation gives $3y''(0) + 1 = 0$, so $y''(0) = -1/3$. The leading coefficient t vanishes at the origin; normalizing by it would produce singular coefficients there.

Step 2: Transform carefully. Writing $c = y''(0)$, the four transforms are

$$-\frac{d}{ds}(s^3Y - s^2 - c), \quad 3(s^2Y - s), \quad -\frac{d}{ds}(sY - 1), \quad Y.$$

Their sum is $-(s^3 + s)Y' - s = 0$. Thus $Y' = -1/(1 + s^2)$ for $s > 0$. Because the transform of a continuous exponential-order function tends to zero as $s \rightarrow \infty$, the constant is fixed:

$$Y(s) = \int_s^\infty \frac{dv}{1 + v^2} = \arctan(1/s), \quad s > 0.$$

Step 3: Invert an integral representation. Since $Y(s) = \int_0^1 s/(s^2 + u^2) du$, absolute convergence for $\operatorname{Re} s > 0$ permits interchange of integrals. Therefore

$$y(t) = \int_0^1 \cos(ut) du = \frac{\sin t}{t} \quad (t > 0), \quad y(0) = 1.$$

For $t > 0$, this function satisfies $ty'' + 2y' + ty = 0$. Differentiating gives the required third-order equation; its analytic extension satisfies the equation at zero too.

Step 4: Prove analytic uniqueness. For $y = \sum_{n \geq 0} c_n t^n$, coefficient matching gives

$$(n+1)((n+2)(n+3)c_{n+2} + c_n) = 0 \quad (n \geq 0).$$

The seeds $c_0 = 1, c_1 = 0$ uniquely give $c_{2k} = (-1)^k/(2k+1)!$ and $c_{2k+1} = 0$, an everywhere convergent series. Ordinary uniqueness on $t > 0$ extends local uniqueness to the whole positive axis. Finally $Y = s^{-1} - (3s^3)^{-1} + (5s^5)^{-1} + O(s^{-7})$, agreeing with $y(0) = 1, y''(0) = -1/3$, and $y^{(4)}(0) = 1/5$ in $Y \sim \sum y^{(j)}(0)/s^{j+1}$.