

Question 5

An unstable homogeneous mode may or may not appear in a forced response. For real parameters a, b , consider

$$y''' + y'' - 2y' = e^{-t}, \quad y(0) = 0, \quad y'(0) = a, \quad y''(0) = b, \quad t \geq 0.$$

Tasks

1. Derive $Y(s)$ and identify all candidate poles before cancellation.
2. Find the necessary and sufficient relation between a, b for a bounded response. Give a complete inverse transform that proves both necessity and sufficiency.
3. Among the bounded responses, find the unique initial pair for which $y(t) \rightarrow 0$. Compute that solution and verify its equation and initial data.
4. An experiment perturbs only b from this special pair to $b + \varepsilon$. Find the exact response error. Explain how an arbitrarily small nonzero error can invalidate boundedness, and state precisely when a final-value calculation is justified.

Question 5 – Solution

Strategy. Inspect the residue of the positive pole before using a final-value theorem.

Step 1: Keep the initial-data numerator. Transformation yields

$$P(s)Y = a(s+1) + b + \frac{1}{s+1}, \quad P(s) = s(s-1)(s+2).$$

The candidate poles are $1, 0, -1, -2$; $\operatorname{Re} s > 1$ is a common convergence half-plane before any cancellation.

Step 2: Classify bounded responses. Residues give

$$y(t) = -\frac{a+b+1}{2} + \frac{2a+b+1/2}{3}e^t + \frac{1}{2}e^{-t} + \frac{-a+b-1}{6}e^{-2t}.$$

Only e^t is unbounded, and no other mode can cancel it asymptotically. Therefore

$$y \text{ is bounded} \iff b = -2a - \frac{1}{2}.$$

For this line of data, $\lim y = (a - 1/2)/2$.

Step 3: Remove the constant mode as well. Zero limit requires $a = 1/2$, $b = -3/2$, giving

$$y_*(t) = \frac{1}{2}(e^{-t} - e^{-2t}).$$

Its first three initial values are $(0, 1/2, -3/2)$. Since $P(-1) = 2$ and $P(-2) = 0$, $P(D)y_* = e^{-t}$, directly checking the equation. Its transform converges for $\operatorname{Re} s > -1$.

Step 4: Quantify sensitivity and theorem use. A change $b \mapsto b + \varepsilon$ gives

$$y - y_* = \varepsilon \left(-\frac{1}{2} + \frac{e^t}{3} + \frac{e^{-2t}}{6} \right).$$

This error has initial state $(0, 0, \varepsilon)$ and is asymptotic to $\varepsilon e^t/3$. Every nonzero perturbation is eventually unbounded. Only after the positive pole cancels do all poles of sY lie strictly left of the imaginary axis; then $\lim y = \lim_{s \rightarrow 0} sY$ is valid. For unstable data, the finite algebraic value at $s = 0$ does not represent a time limit.