

Question 4

Consider the causal, distributional equation

$$y^{(4)} + 2y''' + y'' = 3\delta(t-1) + \delta'(t-1).$$

The response is zero for $t < 0$, with no impulse at the origin. Here δ' is the distributional derivative of δ , and $[v]_1 = v(1+) - v(1-)$ denotes a jump.

Tasks

1. Find the transformed response, using the zero prehistory and both delayed distribution transforms.
2. First invert $H(s) = 1/[s^2(s+1)^2]$, then use differentiation to obtain a real-time formula for y .
3. Derive all four jumps $[y]_1, [y']_1, [y'']_1, [y''']_1$ by matching the coefficients of distributional derivatives. Verify them from your formula.
4. Determine the motion before the impulse, the eventual linear asymptote and the smoothness at switching. Explain why assigning an isolated value at $t = 1$ cannot remove the derivative jump. Sketch the response and its asymptote.

Question 4 – Solution

Strategy. Separate the impulse kernel from the input derivative, then match distributions rather than imposing false continuity conditions.

Step 1: Transform the distributions. For positive delay, $\mathcal{L}\delta(t-1) = e^{-s}$ and $\mathcal{L}\delta'(t-1) = se^{-s}$. Zero prehistory therefore gives

$$Y = e^{-s}(s+3)H(s), \quad H(s) = \frac{1}{s^2(s+1)^2}, \quad \operatorname{Re} s > 0.$$

Step 2: Invert the kernel. Since $H = -2/s + 1/s^2 + 2/(s+1) + 1/(s+1)^2$, $h(r) = r - 2 + (r+2)e^{-r}$ and $h(0) = 0$. Thus sH transforms h' , without an additional origin impulse. For $r = t - 1$,

$$y(t) = u(t-1)k(r), \quad k(r) = 3r - 5 + (2r+5)e^{-r}.$$

Step 3: Match every singular coefficient. Put $J_j = [y^{(j)}]_1$. The coefficients of $\delta''', \delta'', \delta', \delta$ in the left side are, respectively,

$$J_0, \quad J_1 + 2J_0, \quad J_2 + 2J_1 + J_0, \quad J_3 + 2J_2 + J_1.$$

Matching 0, 0, 1, 3 gives $(J_0, J_1, J_2, J_3) = (0, 0, 1, 1)$. Directly, $(k, k', k'', k''')(0) = (0, 0, 1, 1)$; away from $r = 0$, $D^2(D+1)^2k = 0$, completing the distributional verification.

Step 4: Interpret the motion. The response vanishes before $t = 1$ and approaches $3(t-1) - 5$ afterward. It is C^1 , but its one-sided second derivatives differ. Changing a function at one point cannot change those one-sided limits or its distribution.

