

Question 3

Let $u(t - a)$ denote the unit step at a . Solve the delayed-ramp problem

$$(D + 2)^3 y = u(t - 1)(t - 1), \quad (y, y', y'')(0) = (0, 0, 0), \quad t \geq 0.$$

The value assigned to $u(0)$ does not affect an ordinary Laplace transform.

Tasks

1. Transform the initial-value problem, taking care to shift the entire ramp.
2. Obtain a closed form for the response on each side of $t = 1$. Display a partial-fraction decomposition for the unshifted transform.
3. Check the equation on the two open time intervals and determine exactly which derivative first jumps at $t = 1$. State the global smoothness of y .
4. Prove the response is positive after switching, find its large-time linear asymptote, and sketch it together with that asymptote on $t \geq 1$.

Question 3 – Solution

Strategy. Solve a zero-state ramp response, then translate time and check the switching jet.

Step 1: Transform the shifted input. All initial terms vanish. The second shifting theorem gives

$$Y(s) = \frac{e^{-s}}{s^2(s+2)^3}, \quad \operatorname{Re} s > 0.$$

Step 2: Invert before translating. The unshifted fraction decomposes as

$$\frac{1}{s^2(s+2)^3} = -\frac{3}{16s} + \frac{1}{8s^2} + \frac{3}{16(s+2)} + \frac{1}{4(s+2)^2} + \frac{1}{4(s+2)^3}.$$

Thus, with $r = t - 1$,

$$g(r) = \frac{r}{8} - \frac{3}{16} + e^{-2r} \left(\frac{3}{16} + \frac{r}{4} + \frac{r^2}{8} \right), \quad y(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ g(t-1), & t > 1. \end{cases}$$

Step 3: Verify forcing and regularity. The exponential term is annihilated by $(D+2)^3$; applying it to $r/8 - 3/16$ gives r . Also $g(r) = r^4/24 + O(r^5)$ as $r \downarrow 0$. Consequently y, y', y'', y''' match zero at switching, while $y^{(4)}(1+) - y^{(4)}(1-) = 1$. The solution is C^3 but not C^4 ; the original equation also holds at $t = 1$.

Step 4: Establish shape and asymptote. Convolution gives $g(r) = \int_0^r (r-v)v^2 e^{-2v}/2 dv > 0$ for $r > 0$. The exponential correction tends to zero, so the asymptote is $y = (t-1)/8 - 3/16$, approached from above. It is drawn only for $t \geq 1$.

