

Question 2

A sixth-order initial-value problem is

$$(D^2 + 1)^3 y = 0, \quad y^{(j)}(0) = 0 \quad (0 \leq j \leq 4), \quad y^{(5)}(0) = 1,$$

where $D = d/dt$ and $t \geq 0$. To invert a repeated quadratic pole, introduce $H_a(s) = 1/(s^2 + a^2)$ for $a > 0$.

Tasks

1. Expand the differential operator and transform the problem. Explain why only one initial term survives.
2. Starting with $\mathcal{L}^{-1}H_a = \sin(at)/a$, use differentiation with respect to a twice to invert $(s^2 + a^2)^{-3}$. Then set $a = 1$.
3. Verify the equation and initial state without relying on the transform calculation. Identify the first nonzero Taylor term.
4. Describe the pole multiplicities and a convergence half-plane. Decide whether the solution is bounded and whether the final-value theorem can be used, giving a decisive argument for each claim.

Question 2 – Solution

Strategy. Parameter differentiation resolves repeated quadratic poles while a Taylor check tests the high-order initial data.

Step 1: Transform the sixth-order problem. The equation is $y^{(6)} + 3y^{(4)} + 3y'' + y = 0$. Only $\mathcal{L}(y^{(6)}) = s^6 Y - 1$ has a nonzero initial subtraction, so

$$Y(s) = \frac{1}{(s^2 + 1)^3}.$$

Step 2: Differentiate a transform family. Since $\partial_a(s^2 + a^2)^{-m} = -2ma(s^2 + a^2)^{-m-1}$,

$$h_2(t) = \frac{\sin(at) - at \cos(at)}{2a^3}, \quad h_3(t) = -\frac{1}{4a} \partial_a h_2(t) = \frac{3 \sin(at) - 3at \cos(at) - a^2 t^2 \sin(at)}{8a^5}.$$

Differentiation under the integral is valid for a in a compact positive interval and $\operatorname{Re} s > 0$, since the derivatives have polynomial growth. Hence

$$y(t) = \frac{3 \sin t - 3t \cos t - t^2 \sin t}{8}.$$

Step 3: Verify independently. Each term is the real or imaginary part of e^{it} times a polynomial of degree at most two; $(D - i)^3$ annihilates these complex modes, and the conjugate factor annihilates their conjugates. Thus $(D^2 + 1)^3 y = 0$. Expanding the displayed trigonometric expression gives

$$y(t) = \frac{t^5}{120} - \frac{t^7}{1680} + O(t^9).$$

This proves the five zero initial derivatives and $y^{(5)}(0) = 1$.

Step 4: Distinguish oscillation from boundedness. There are triple poles at $s = \pm i$; $\operatorname{Re} s > 0$ gives absolute convergence. Along $t_n = \pi/2 + 2\pi n$, $y(t_n) = (3 - t_n^2)/8 \rightarrow -\infty$, so the response is unbounded. The poles of sY on the imaginary axis invalidate the final-value theorem. The formal value $\lim_{s \rightarrow 0} sY = 0$ is not a time limit.