

Question 1

Consider the initial-value problem, for $t \geq 0$,

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0, \quad (y, y', y'', y''')(0) = (1, 0, 0, 0).$$

A student writes $(s+1)^4 Y(s) = s^3$ and concludes that the initial displacement can be represented by a single term from the fourth derivative.

Tasks

1. Derive the correct transformed equation, displaying the initial terms contributed by every derivative. Identify the student's error.
2. Find $Y(s)$ and invert it by organizing the numerator in powers of $s + 1$.
3. Verify the differential equation and all four initial values directly. Determine whether the response ever increases or changes sign.
4. Find a half-plane of convergence and the initial and final limits. Explain why a quadruple pole does not prevent decay, and sketch the response.

Question 1 – Solution

Strategy. Collect every initial contribution before exploiting the repeated shifted pole.

Step 1: Transform all derivatives. The derivative transforms are $s^4Y - s^3$, $s^3Y - s^2$, $s^2Y - s$, and $sY - 1$. Thus

$$(s+1)^4Y = s^3 + 4s^2 + 6s + 4.$$

The student omitted the initial terms from $4y''' + 6y'' + 4y'$.

Step 2: Use shifted powers. Writing $z = s + 1$ makes the numerator $z^3 + z^2 + z + 1$. Therefore

$$Y = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}, \quad \boxed{y(t) = e^{-t} \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} \right)}.$$

Step 3: Check the state and shape. For $y = e^{-t}p(t)$, $(D+1)^4y = e^{-t}p^{(4)} = 0$. Moreover $y' = -t^3e^{-t}/6$, so $y'(0) = y''(0) = y'''(0) = 0$ and $y(0) = 1$. Every term in p is positive on $t \geq 0$; hence $y > 0$ and $y' < 0$ for $t > 0$.

Step 4: Check convergence and limits. The transform converges absolutely for $\operatorname{Re} s > -1$. Also $\lim_{s \rightarrow \infty} sY = 1$ and $\lim_{s \rightarrow 0} sY = 0$. The final-value theorem applies because all poles of sY lie at -1 . Polynomial growth from multiplicity is dominated by e^{-t} , so $y(t) \rightarrow 0$.

