

Question 10

For the zero-data IVP

$$y''' + y' = e^x, \quad y(0) = y'(0) = y''(0) = 0,$$

a proposed approximation on $[0, 1]$ is $q(x) = x^3/6 + x^4/24$.

Tasks

1. Compute the residual $r = q''' + q' - e^x$ and check the approximation's initial data. Express the residual using the remainder after the cubic Taylor polynomial of e^x .
2. Use variation of parameters to represent the error $E = y - q$ by a definite integral with zero initial data. Prove whether the approximation lies above or below the true solution on $(0, 1]$.
3. Use Taylor's remainder theorem and $1 - \cos s \leq s^2/2$ to prove $0 \leq E(x) \leq ex^7/5040$ for $0 \leq x \leq 1$. Decide whether an absolute-error tolerance of 10^{-3} is certified throughout this interval.
4. Independently verify $Y(x) = (e^x + \cos x - \sin x)/2 - 1$ as the exact IVP solution. Compare its error at $x = 1$ with the certified upper bound, and explain why direct agreement at finitely many sample points is weaker than the integral certificate.

Question 10 – Solution

Strategy. Treat the residual as an error forcing and bound its full response, not just sampled values.

Step 1: Compute the forcing error. We have $q''' = 1 + x$ and $q' = x^2/2 + x^3/6$, so

$$r = 1 + x + x^2/2 + x^3/6 - e^x = -R_4(x),$$

where $R_4 = e^x - (1 + x + x^2/2 + x^3/6)$. The values $q(0), q'(0), q''(0)$ are zero. Thus $E = y - q$ has a zero initial vector and satisfies $E''' + E' = R_4$.

Step 2: Derive the positive error integral. For the basis $1, \cos x, \sin x$, the variation derivatives for this error equation are $(R_4, -R_4 \cos x, -R_4 \sin x)$. Integrating from zero gives

$$E(x) = \int_0^x [1 - \cos(x-t)] R_4(t) dt.$$

For $0 < t < x \leq 1$, both factors are strictly positive. Hence $E(x) > 0$ on $(0, 1]$: the approximation underestimates the true solution.

Step 3: Certify a uniform error bound. Taylor's theorem gives $0 \leq R_4(t) \leq et^4/24$ on $[0, 1]$. Therefore

$$0 \leq E(x) \leq \frac{e}{48} \int_0^x (x-t)^2 t^4 dt = \boxed{\frac{ex^7}{5040}},$$

where the polynomial integral equals $x^7/105$. The maximum certified error is $e/5040 \approx 0.00053934 < 10^{-3}$, so the tolerance holds throughout $[0, 1]$.

Step 4: Compare with an exact independent check. For $Y = (e^x + \cos x - \sin x)/2 - 1$, direct differentiation gives $Y''' + Y' = e^x$ and its initial vector is zero. Uniqueness therefore implies $Y = y$. At 1, $q(1) = 5/24$, so the actual error is

$$E(1) = \frac{e + \cos 1 - \sin 1}{2} - \frac{29}{24}.$$

Numerically, $E(1) \approx 0.00022324$, positive and smaller than the certified bound. Exact evaluation or sample agreement is useful corroboration, but finitely many values cannot exclude larger errors between them. The integral inequality proves the bound for every point of the interval.

