

**Question 9**

On  $x > 0$ , consider

$$x^4 y^{(4)} = 1, \quad y(1) = y'(1) = y''(1) = y'''(1) = 0.$$

A proposed response kernel for  $x \geq t > 0$  is

$$\hat{K}(x, t) = \frac{(x-t)^3}{6x^4}.$$

The placement of the leading-coefficient factor must be checked, not guessed.

**Tasks**

1. Normalize the equation and derive the correct kernel from variation of parameters using  $1, x, x^2/2, x^3/6$ .
2. For fixed  $t$ , verify the correct kernel's homogeneous equation when  $x > t$  and its first four diagonal data, through the third  $x$ -derivative.
3. Integrate the correct kernel from 1 to  $x$  to find the IVP solution in elementary form. Verify its initial data and forcing.
4. Show that  $\hat{K}$  has the same diagonal data but is still wrong. Compute the response it would predict and evaluate  $x^4 \hat{y}^{(4)}$  at  $x = 2$ . Explain which additional kernel requirement fails.

## Question 9 – Solution

**Strategy.** Evaluate the normalized forcing at the integration variable and check the kernel away from its diagonal.

**Step 1: Derive the correctly normalized kernel.** The equation is  $y^{(4)} = x^{-4}$ . For the given basis, variation of parameters gives  $u' = x^{-4}(-x^3/6, x^2/2, -x, 1)^T$ . Integrating from 1 and combining yields

$$K(x, t) = \frac{(x-t)^3}{6t^4}, \quad y(x) = \int_1^x K(x, t) dt.$$

The denominator is evaluated at the source variable  $t$ . The same polynomial expression and oriented integral apply for  $0 < x < 1$ .

**Step 2: Verify both kernel requirements.** For fixed  $t > 0$ ,  $K$  is a cubic polynomial in  $x$ , so  $x^4 K_{xxxx} = 0$  for  $x > t$ . Its diagonal data are

$$K(t, t) = K_x(t, t) = K_{xx}(t, t) = 0, \quad K_{xxx}(t, t) = 1/t^4.$$

Leibniz differentiation gives  $y^{(4)}(x) = 1/x^4$ , because the fourth kernel derivative is zero and the third diagonal derivative supplies exactly that term.

**Step 3: Evaluate and check the elementary response.** Expanding  $(x-t)^3/t^4$  and integrating its four powers of  $t$  gives

$$y = \frac{x^3}{18} - \frac{x^2}{4} + \frac{x}{2} - \frac{\ln x}{6} - \frac{11}{36}.$$

Its derivatives are  $y' = x^2/6 - x/2 + 1/2 - 1/(6x)$ ,  $y'' = x/3 - 1/2 + 1/(6x^2)$ , and  $y''' = 1/3 - 1/(3x^3)$ . These and  $y$  vanish at 1; one further derivative gives  $y^{(4)} = x^{-4}$ .

**Step 4: Expose the misleading diagonal agreement.** The factor  $(x-t)^3$  ensures that  $\hat{K}$  has the same three zero diagonal values and  $\hat{K}_{xxx}(t, t) = 1/t^4$ . But its proposed response is

$$\hat{y} = \frac{1}{6x^4} \int_1^x (x-t)^3 dt = \frac{(x-1)^4}{24x^4}.$$

It satisfies the zero initial data, yet

$$x^4 \hat{y}^{(4)} = -\frac{4}{x} + \frac{30}{x^2} - \frac{60}{x^3} + \frac{35}{x^4}, \quad x^4 \hat{y}^{(4)} \Big|_{x=2} = \boxed{3/16 \neq 1}.$$

The missing requirement is the homogeneous equation in  $x$  away from  $x = t$ . Correct diagonal data alone do not make a valid response kernel.