

Question 8

Let g be continuous and nonnegative on $[0, \infty)$, and consider

$$y''' + y' = g(x), \quad y(0) = y'(0) = y''(0) = 0.$$

Use the homogeneous basis $1, \cos x, \sin x$.

Tasks

1. Derive the variation parameters and combine them into a single response integral.
2. Prove that $y(x) \geq 0$ for every $x \geq 0$. Give a sufficient condition for strict positivity at a specified $x > 0$ and justify it analytically.
3. For the continuous pulse $g(x) = \sin^2 x$ on $[0, \pi]$ and $g(x) = 0$ for $x \geq \pi$, compute $y(x)$ for all $x \geq \pi$ using the integrals of g , $g \cos x$ and $g \sin x$.
4. Determine the exact minimum and maximum of this post-pulse response. Prove that a nonnegative forcing need not make y monotone, even though the solution stays nonnegative.

Question 8 – Solution

Strategy. Distinguish a nonnegative response kernel from the sign of its derivative.

Step 1: Compute the response kernel. The derivative matrix of $1, \cos x, \sin x$ has Wronskian 1. Its parameter equations give

$$u'_0 = g, \quad u'_c = -g \cos x, \quad u'_s = -g \sin x.$$

Integrating from zero and combining by the cosine difference identity yields

$$y(x) = \int_0^x [1 - \cos(x-t)]g(t) dt.$$

The kernel and its first derivative vanish at zero, its second derivative is 1 there, and its third plus first derivative vanishes. These facts verify the equation and zero data.

Step 2: Prove positivity without a monotonicity assumption. Both factors in the integrand are nonnegative. If $g(t_0) > 0$ for some $0 \leq t_0 < x$, continuity gives an interval of positive forcing inside $[0, x)$. The zeros of $1 - \cos(x-t)$ are isolated, so the integral is strictly positive. In particular, a continuous nonnegative forcing that is not identically zero on $[0, x)$ suffices.

Step 3: Evaluate the post-pulse moments. For the stated pulse,

$$\int_0^\pi \sin^2 t dt = \frac{\pi}{2}, \quad \int_0^\pi \sin^2 t \cos t dt = 0, \quad \int_0^\pi \sin^3 t dt = \frac{4}{3}.$$

The first follows from the double-angle identity, the second from the derivative of $\sin^3 t/3$, and the third by substituting $u = \cos t$. Thus

$$y(x) = \frac{\pi}{2} - \frac{4}{3} \sin x \quad (x \geq \pi).$$

For plotting the initial interval, evaluating the same kernel gives $y = x/2 - 2 \sin x/3 + \sin 2x/12$ on $[0, \pi]$; its state matches the displayed branch at π .

Step 4: Separate positivity from monotonicity. The post-pulse range is

$$\frac{\pi}{2} - \frac{4}{3} \leq y(x) \leq \frac{\pi}{2} + \frac{4}{3}.$$

Both bounds are attained infinitely often; the lower bound is positive. Yet $y' = -4 \cos x/3$ changes sign, and $y'(2\pi) = -4/3 < 0$. Thus the nonnegative input gives a nonnegative response that continues oscillating after the pulse. Positivity of $1 - \cos s$ does not imply positivity of its derivative $\sin s$.

