

Question 7

For the variable-coefficient IVP

$$y''' - xy'' = 1, \quad y(0) = y'(0) = y''(0) = 0,$$

define $F(x) = \int_0^x (x-t)e^{t^2/2} dt$ and use the homogeneous basis $1, x, F(x)$.

Tasks

1. Verify the basis and its Wronskian, then derive the three variation-parameter derivatives.
2. Combine the lower-limit-zero parameter integrals into $y(x) = \int_0^x K(x,t) dt$. Write K both using F, F' and as a definite integral for $x \geq t$.
3. Differentiate to verify the IVP. Prove that y is strictly convex and $y(x) > x^3/6$ for every $x > 0$.
4. Determine whether $K(x,t)$ can depend only on $x-t$. Use a derivative of the kernel to give a precise obstruction to treating this variable-coefficient response as a translation-invariant convolution.

Question 7 – Solution

Strategy. Exploit the triangular derivative matrix while retaining both kernel variables.

Step 1: Solve the parameter equations. We have $F(0) = F'(0) = 0$, $F'' = e^{x^2/2}$ and $F''' = xF''$. Thus $1, x, F$ solve the homogeneous equation and have Wronskian $e^{x^2/2} > 0$. The triangular system gives

$$u_2' = e^{-x^2/2}, \quad u_1' = -F'e^{-x^2/2}, \quad u_0' = (xF' - F)e^{-x^2/2}.$$

Step 2: Combine the parameter integrals. Multiplying by $1, x, F(x)$ and integrating from zero yields

$$K(x, t) = e^{-t^2/2} [F(x) - F(t) - (x - t)F'(t)].$$

Taylor's integral identity, obtained by integrating F'' twice, gives for $x \geq t$

$$K(x, t) = e^{-t^2/2} \int_t^x (x - s)e^{s^2/2} ds, \quad y(x) = \int_0^x K(x, t) dt.$$

The factor involving t is essential; it comes from the inverse derivative matrix evaluated at the integration variable.

Step 3: Verify the equation and inequalities. At the diagonal, $K(t, t) = K_x(t, t) = 0$, while

$$K_{xx}(x, t) = e^{(x^2-t^2)/2}, \quad y''(x) = e^{x^2/2} \int_0^x e^{-t^2/2} dt.$$

Hence $y''' = xy'' + 1$ and all three initial data vanish. For $x > 0$, $y'' > 0$, proving strict convexity. Also $y''' = 1 + xy'' > 1$ there. Integrating this strict inequality three times from the zero initial vector gives $y(x) > x^3/6$ for every $x > 0$.

Step 4: Test translation invariance. If $K(x, t) = H(x - t)$, then $K_{xx} = H''(x - t)$ would have the same value at all pairs with the same difference. But for any fixed $h > 0$,

$$K_{xx}(t + h, t) = e^{th+h^2/2},$$

which varies with t . Therefore this kernel is not a function of $x - t$ alone. The variable coefficient distinguishes the absolute source time from the elapsed time.