

Question 5

On the positive half-line, consider

$$y''' = \frac{1}{x}, \quad y(1) = y'(1) = y''(1) = 0.$$

Use the homogeneous basis $1, x, x^2/2$ and integrate the variation parameters from 1.

Tasks

1. Derive the parameter derivatives and compute the complete IVP solution.
2. Verify the original equation and all initial data. Find the one-sided limits of y, y', y'' as $x \rightarrow 0^+$.
3. Determine whether the solution admits a C^1 or C^2 extension to the endpoint 0. Explain why a divergent parameter does not necessarily force y itself to diverge.
4. Test the attempted lower-limit-zero formula $\frac{1}{2} \int_0^x (x-t)^2/t \, dt$. Decide whether it converges for fixed $x > 0$, and explain why finite endpoint values of y, y' do not justify using zero as a regular IVP base point.

Question 5 – Solution

Strategy. Separate parameter divergence, solution limits and the regularity needed for initial data.

Step 1: Integrate from the regular point. The triangular parameter system for $y''' = g$ gives $(u'_0, u'_1, u'_2) = (x^2 g/2, -xg, g)$. Here $g = 1/x$, so

$$u_0 = (x^2 - 1)/4, \quad u_1 = 1 - x, \quad u_2 = \ln x.$$

Combining $u_0 + xu_1 + x^2 u_2/2$ yields

$$y = \frac{x^2 \ln x}{2} - \frac{3x^2}{4} + x - \frac{1}{4} \quad (x > 0).$$

Step 2: Check derivatives and endpoint behavior. Direct differentiation gives

$$y' = x \ln x - x + 1, \quad y'' = \ln x, \quad y''' = 1/x.$$

These formulas and $\ln 1 = 0$ verify all three data at 1. Since $x \ln x$ and $x^2 \ln x$ tend to zero at the positive endpoint,

$$\lim_{x \rightarrow 0^+} y = -1/4, \quad \lim_{x \rightarrow 0^+} y' = 1, \quad \lim_{x \rightarrow 0^+} y'' = -\infty.$$

Step 3: Classify the extension. Defining $y(0) = -1/4$ gives right derivative 1, because $(y(x) + 1/4)/x \rightarrow 1$. This derivative is continuous from the right, so the solution extends as a C^1 function to $[0, \infty)$; it may also be continued to the left with matching value and slope. No C^2 extension is possible because y'' diverges. Although $u_2 = \ln x$ diverges, it is multiplied by $x^2/2$ in y , whose contribution tends to zero.

Step 4: Reject the singular base point. For fixed $x > 0$ and $0 < t < x/2$, $(x - t)^2/(2t) \geq x^2/(8t)$. The proposed improper integral therefore diverges to $+\infty$. Its lower limit cannot simply be moved to zero. The equation is singular there, and even the solution's second derivative lacks finite initial data. Two finite endpoint limits do not supply a regular third-order initial state.

