

Question 3

On $x > 0$, consider the Euler-type equation

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x.$$

Use the proposed homogeneous basis x, x^2, x^3 and variation of parameters, rather than a guessed logarithmic trial.

Tasks

1. Verify the three homogeneous solutions and their Wronskian. Normalize the equation and write the parameter system with the correct right-hand side.
2. Solve for the parameter derivatives and integrate to obtain one particular solution. Write the complete general solution.
3. Impose $y(1) = y'(1) = y''(1) = 0$. Verify all three initial data and the forcing directly.
4. State the interval containing 1 on which this construction and the standard linear IVP theorem apply. Explain why evaluating the same parameter system at $x = 0$ is invalid, even if some expressions have finite limits there.

Question 3 – Solution

Strategy. Normalize the Euler operator before solving the derivative equations, and retain its positive-domain restriction.

Step 1: Verify the basis and normalization. For $y = x^m$, the left side is $(m-1)(m-2)(m-3)x^m$. Thus the proposed functions are homogeneous. Their Wronskian is $2x^3 \neq 0$ for $x > 0$. Division by x^3 gives normalized forcing $x \ln x$, so

$$\begin{pmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{pmatrix} \begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x \ln x \end{pmatrix}.$$

Step 2: Integrate the solved parameter derivatives. Elimination gives $u'_1 = x^2 \ln x / 2$, $u'_2 = -x \ln x$, $u'_3 = \ln x / 2$. Convenient antiderivatives are

$$u_1 = \frac{x^3 \ln x}{6} - \frac{x^3}{18}, \quad u_2 = -\frac{x^2 \ln x}{2} + \frac{x^2}{4}, \quad u_3 = \frac{x \ln x}{2} - \frac{x}{2}.$$

Combining $xu_1 + x^2u_2 + x^3u_3$ gives

$$y_p = x^4 \left(\frac{\ln x}{6} - \frac{11}{36} \right), \quad y = y_p + c_1x + c_2x^2 + c_3x^3.$$

Integration constants are already represented by the homogeneous terms.

Step 3: Fit and check the initial vector. The particular solution has vector $(-11/36, -19/18, -5/2)$ at 1. The three linear equations for its cancellation give

$$y = \frac{x}{18} - \frac{x^2}{4} + \frac{x^3}{2} + x^4 \left(\frac{\ln x}{6} - \frac{11}{36} \right).$$

The added terms have vector $(11/36, 19/18, 5/2)$, verifying the data. For a direct residual check, let $\theta = xD$. The operator is $(\theta-1)(\theta-2)(\theta-3)$; on $x^4(a \ln x + b)$ it gives $x^4(6a \ln x + 11a + 6b)$. With $a = 1/6$, $b = -11/36$, this is exactly $x^4 \ln x$.

Step 4: Respect the regular interval. The normalized coefficients, forcing and nonsingular derivative matrix are valid throughout $(0, \infty)$, the largest such interval containing 1. At zero the leading coefficient and Wronskian vanish and $\ln x$ is undefined. Finite limits of selected terms do not make that parameter system invertible or extend the theorem across zero.