

Question 2

For $x \geq 0$, consider

$$y''' - y' = e^{-x^2}.$$

Use the homogeneous basis $1, e^x, e^{-x}$; definite integrals may be left unevaluated.

Tasks

1. Verify the basis and derive the three parameter derivatives by variation of parameters.
2. Find the solution y_0 with $y_0(0) = y_0'(0) = y_0''(0) = 0$, expressed as one integral. Verify its kernel and initial data.
3. Determine $\lim_{x \rightarrow \infty} e^{-x} y_0(x)$ and prove that it is positive. Justify convergence of every improper integral used.
4. Keep $y(0) = y'(0) = 0$ but choose $y''(0)$ so that the response is bounded on $[0, \infty)$. Prove uniqueness of this choice and determine the finite limit of the resulting solution.

Question 2 – Solution

Strategy. Use parameter integrals to isolate the growing homogeneous mode and the initial curvature needed to cancel it.

Step 1: Solve the derivative system. The functions solve $y''' - y' = 0$ and have Wronskian 2. With $g = e^{-x^2}$, the equations are

$$u'_0 + e^x u'_+ + e^{-x} u'_- = 0, \quad e^x u'_+ - e^{-x} u'_- = 0, \quad e^x u'_+ + e^{-x} u'_- = g.$$

Thus $u'_0 = -g$, $u'_+ = e^{-x}g/2$, and $u'_- = e^xg/2$.

Step 2: Construct the zero-data response. Integrating from zero and combining gives

$$y_0(x) = \int_0^x [\cosh(x-t) - 1] e^{-t^2} dt.$$

For $G(s) = \cosh s - 1$, we have $G(0) = G'(0) = 0$, $G''(0) = 1$, and $G''' - G' = 0$. Leibniz differentiation therefore gives $y_0''' - y_0' = g$ and the three zero initial data.

Step 3: Identify the growing coefficient. Define

$$A(x) = \int_0^x e^{-t^2-t} dt, \quad B(x) = \int_0^x e^{-t^2+t} dt, \quad C(x) = \int_0^x e^{-t^2} dt.$$

Their limits $A_\infty, B_\infty, C_\infty$ are finite: for $t \geq 2$, each integrand is at most $e^{-t^2/2} \leq e^{-t}$. All are positive. Since $y_0 = e^x A(x)/2 + e^{-x} B(x)/2 - C(x)$,

$$\lim_{x \rightarrow \infty} e^{-x} y_0(x) = A_\infty/2 > 0.$$

Step 4: Cancel growth without losing the first two data. Any other solution with the same value and slope is $y = y_0 + k(\cosh x - 1)$, where $k = y''(0)$. Boundedness requires $k = -A_\infty$, and no other choice cancels the growing coefficient. For this choice,

$$y = -\frac{1}{2}e^x(A_\infty - A(x)) + \frac{1}{2}e^{-x}(B(x) - A_\infty) + A_\infty - C(x).$$

The first term tends to zero: for $x > 0$, $e^x \int_x^\infty e^{-t^2-t} dt \leq e^{-x^2}$. The second also tends to zero. Therefore the choice is sufficient, and

$$y''(0) = -A_\infty, \quad \lim_{x \rightarrow \infty} y = A_\infty - C_\infty.$$