

Question 8

Consider the initially resting response

$$(D^2 + 1)(D^2 + 9)y = \cos^3 x, \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

A nonlinear expression in the independent variable can conceal several resonant harmonics while the differential equation remains linear in y .

Tasks

1. Expand the forcing into elementary harmonics. Identify all resonances and select a complete real particular-solution trial.
2. Determine a particular solution by coefficient matching, and write the full homogeneous correction.
3. Impose the initial data and verify the resulting response, including the forcing and the initial curvature.
4. Evaluate the IVP solution at $x_n = \pi/2 + 2\pi n$, $n = 0, 1, \dots$, to prove unboundedness. Derive an explicit envelope for its magnitude valid for every $x \geq 0$.

Question 8 – Solution

Strategy. Expand the trigonometric product before choosing resonance factors.

Step 1: Expose both forcing frequencies. The identity $\cos^3 x = (3 \cos x + \cos 3x)/4$ reveals simple resonance at both $\pm i$ and $\pm 3i$. A full trial is

$$x(A \cos x + B \sin x + C \cos 3x + D \sin 3x).$$

There is no nonlinear dependence on the unknown y ; the right-hand side is a known function of x .

Step 2: Match each harmonic. For $L = (D^2 + 1)(D^2 + 9)$, differentiation gives $L(x \sin x) = 16 \cos x$ and $L(x \sin 3x) = -48 \cos 3x$. The cosine-weighted partners generate sine forcing and therefore have zero coefficients. Thus

$$y_p = \frac{3x \sin x}{64} - \frac{x \sin 3x}{192}, \quad y_h = a \cos x + b \sin x + c \cos 3x + d \sin 3x.$$

The residual is $3 \cos x/4 + \cos 3x/4 = \cos^3 x$.

Step 3: Correct the initial curvature. The particular term is even, has value zero, and has $y_p''(0) = 3/32 - 1/32 = 1/16$. The zero data require $a + c = 0$, $-a - 9c = -1/16$, and $b + 3d = -b - 27d = 0$. Hence

$$y = \frac{x(9 \sin x - \sin 3x)}{192} + \frac{\cos 3x - \cos x}{128}.$$

Here $a = -1/128, c = 1/128, b = d = 0$. The cosine correction has second derivative $-1/16$ at zero and cancels the particular curvature; evenness verifies the odd initial derivatives.

Step 4: Prove growth and a global envelope. At $x_n = \pi/2 + 2\pi n$, the sine values are 1, -1 and both cosines vanish, so $y(x_n) = 5x_n/96 \rightarrow \infty$. Also $9 \sin x - \sin 3x = 6 \sin x + 4 \sin^3 x$, whose absolute value is at most 10. Therefore

$$|y(x)| \leq \frac{5x}{96} + \frac{1}{64} \quad (x \geq 0).$$

The displayed envelopes are bounds, not assertions that equality holds at every point.

