

Question 7

For real parameters a, b, c , consider

$$y^{(4)} + y'' = a + b \cos x + c \cos 2x.$$

A forcing with zero average over a period need not admit a periodic response.

Tasks

1. Use undetermined coefficients to find a particular solution for all a, b, c , and write the full real general solution.
2. Give necessary and sufficient conditions on the parameters and homogeneous constants for boundedness on $[0, \infty)$.
3. Determine exactly when the equation admits a 2π -periodic solution. Verify the necessary restrictions independently by integrating the equation against 1 and $\cos x$ over one period.
4. Give a forcing in this family with zero average but no periodic solution, and another nonzero forcing that does admit periodic solutions. Explain the different obstructions caused by a nonzero mean and by a resonant harmonic.

Question 7 – Solution

Strategy. Track the zero-frequency resonance and the oscillatory resonance separately.

Step 1: Calculate the resonant and nonresonant pieces. The characteristic polynomial is $r^2(r^2 + 1)$. Direct substitution gives

$$L(x^2/2) = 1, \quad L(x \sin x) = -2 \cos x, \quad L(\cos 2x) = 12 \cos 2x.$$

Consequently,

$$y = \frac{ax^2}{2} - \frac{b}{2}x \sin x + \frac{c}{12} \cos 2x + k_0 + k_1x + A \cos x + B \sin x.$$

The factors x^2 and x reflect the double root at zero and simple roots at $\pm i$, respectively.

Step 2: Classify boundedness exactly. If $a \neq 0$, division by x^2 gives a nonzero limit, so boundedness fails. Once $a = 0$, a bounded solution would satisfy

$$0 = \lim_{x \rightarrow \infty} \frac{y(x)}{x}, \quad \frac{y(x)}{x} = k_1 - \frac{b}{2} \sin x + o(1).$$

Along sequences with sine equal to 1 and -1 , this requires both $k_1 - b/2 = 0$ and $k_1 + b/2 = 0$. Hence

$$\text{bounded on } [0, \infty) \iff a = b = k_1 = 0.$$

These conditions suffice because all remaining terms are bounded.

Step 3: Check periodic compatibility independently. With $a = b = 0$, choosing $k_1 = 0$ gives a 2π -periodic solution for every c . Conversely, for a 2π -periodic C^4 solution, integration gives

$$\int_0^{2\pi} (y^{(4)} + y'') dx = 0 = 2\pi a.$$

Integration by parts transfers the even derivatives onto $\cos x$ without boundary terms. Since $(D^4 + D^2) \cos x = 0$, orthogonality then gives $0 = b\pi$. Thus $a = b = 0$ is necessary and sufficient for existence of a 2π -periodic response.

Step 4: Explain why zero mean is insufficient. The forcing $\cos x$ has zero average but produces the unavoidable term $-x \sin x/2$, so no solution is periodic. In contrast, $\cos 2x$ admits $y = \cos 2x/12$. A nonzero mean excites quadratic drift through the double zero root; a first harmonic excites linear oscillatory growth. Removing only the mean addresses just one of these two resonances.