

Question 6

Let $\omega \geq 0$ and consider the zero-data response

$$(D^2 + 1)(D^2 + 4)y_\omega = \cos(\omega x), \quad y_\omega(0) = y'_\omega(0) = y''_\omega(0) = y'''_\omega(0) = 0.$$

Tasks

1. For $\omega \neq 1, 2$, find a particular solution and then the full zero-data response by adding homogeneous modes.
2. Solve the IVP separately at each resonant frequency $\omega = 1$ and $\omega = 2$, using resonance-corrected trials.
3. Rewrite the nonresonant answer using differences of cosines and prove that it tends to the resonant answer as $\omega \rightarrow 1$ for every fixed x .
4. Classify which zero-data responses are bounded on $[0, \infty)$. Explain why the fixed- x limit in the previous task does not provide a bound uniform over all $x \geq 0$ as $\omega \rightarrow 1$.

Question 6 – Solution

Strategy. Enforce the initial data before taking a resonant limit; separate divergent terms can cancel.

Step 1: Solve away from resonance. Set $K = [(1 - \omega^2)(4 - \omega^2)]^{-1}$. The trial $K \cos \omega x$ has the required residual. Odd initial derivatives vanish; adding $A \cos x + C \cos 2x$ and matching the value and second derivative gives

$$y_\omega = K \cos \omega x - \frac{\cos x}{3(1 - \omega^2)} + \frac{\cos 2x}{3(4 - \omega^2)}, \quad \omega \neq 1, 2.$$

Indeed $A + C = -K$ and $A + 4C = -K\omega^2$, which verify the remaining two data.

Step 2: Solve at the two simple resonances. For $L = (D^2 + 1)(D^2 + 4)$, direct differentiation gives $L(x \sin x) = 6 \cos x$ and $L(x \sin 2x) = -12 \cos 2x$. Adding homogeneous cosine terms to enforce zero value and curvature yields

$$y_1 = \frac{x \sin x}{6} + \frac{\cos 2x - \cos x}{9}, \quad y_2 = -\frac{x \sin 2x}{12} + \frac{\cos x - \cos 2x}{9}.$$

Both are even and vanish at zero. Their second derivatives at zero vanish by cancellation of $1/3$ with $-1/3$ in each case, so all four initial data hold.

Step 3: Take the limit of the combined response. Partial fractions give

$$y_\omega = \frac{1}{3} \left(\frac{\cos \omega x - \cos x}{1 - \omega^2} - \frac{\cos \omega x - \cos 2x}{4 - \omega^2} \right).$$

For fixed x , the first ratio tends to $x \sin x/2$ by differentiating numerator and denominator with respect to ω . The second tends to $(\cos x - \cos 2x)/3$. The resulting limit is exactly y_1 .

Step 4: Distinguish the two limit questions. Every fixed nonresonant response is a bounded sum of cosines. At resonance, the linear sine term has unbounded values along its phase maxima, and the added cosines remain bounded. Thus boundedness holds exactly when $\omega \notin \{1, 2\}$. A common bound for all $x \geq 0$ and all nonresonant ω sufficiently near 1 would pass to the fixed- x limit and bound y_1 everywhere, a contradiction.

