

Question 5

Consider

$$(D + 1)^2(D - 2)y = e^{-x}.$$

An annihilator can simplify the forcing while introducing extra solutions that must later be rejected.

Tasks

1. Apply a minimal constant-coefficient annihilator of the forcing and write the full solution of the resulting fourth-order homogeneous equation.
2. Substitute that enlarged family into the original third-order equation. Find the constraint that removes the extra freedom and write the actual general solution.
3. Find the solution with $y(0) = y'(0) = y''(0) = 0$. Determine the coefficient of its growing exponential and its limit after multiplication by e^{-2x} .
4. Instead require $y(0) = y'(0) = 0$ and decay as $x \rightarrow +\infty$. Find this solution and the initial value $y''(0)$ it requires. Explain why decaying forcing does not guarantee that the zero-data response decays.

Question 5 – Solution

Strategy. Retain the original equation as a constraint after annihilating its right-hand side.

Step 1: Solve the enlarged equation. The forcing is annihilated by $D + 1$. Applying it gives $(D + 1)^3(D - 2)y = 0$, whose general solution is

$$y = e^{-x}(A + Bx + Cx^2) + Ee^{2x}.$$

This family has four constants, although the original equation has order three.

Step 2: Restore the original forcing. Under $y = e^{-x}v$, the original operator becomes $e^{-x}D^2(D - 3)v$. For $v = A + Bx + Cx^2$ its value is $-6Ce^{-x}$; the Ee^{2x} term is homogeneous. Thus $C = -1/6$, leaving

$$y = e^{-x}(A + Bx - x^2/6) + Ee^{2x}.$$

Annihilation alone only ensures that the original residual is a multiple of e^{-x} , not that its coefficient equals one.

Step 3: Fit the zero initial vector. The three equations are $A + E = 0$, $B - A + 2E = 0$, and $A - 2B - 1/3 + 4E = 0$. They yield $E = 1/27$, $A = -1/27$, $B = -1/9$, so

$$y_0 = \frac{e^{2x}}{27} - e^{-x} \left(\frac{1}{27} + \frac{x}{9} + \frac{x^2}{6} \right), \quad \lim_{x \rightarrow \infty} e^{-2x} y_0 = \frac{1}{27}.$$

The displayed coefficient equations verify the initial data, and Step 2 verifies the forcing.

Step 4: Replace curvature data by a future condition. Decay requires $E = 0$; with $y(0) = y'(0) = 0$, this gives $A = B = 0$. Hence

$$y_d = -x^2 e^{-x}/6, \quad y_d''(0) = -1/3.$$

This is the unique decaying response with those first two data. The zero-data response must add a growing homogeneous mode to cancel the particular solution's initial curvature. Decay of the input alone does not suppress that mode.

