

Question 4

On $\mathbb{R}$, consider

$$y^{(4)} - y = 6x^2 + 8e^{-x} + 10 \cos x.$$

Its three forcing terms require different coefficient calculations.

Tasks

1. Factor the characteristic polynomial, write the full real homogeneous solution, and select a separate particular-solution trial for each forcing term.
2. Determine all coefficients and assemble one particular solution.
3. Verify each forcing contribution directly using $L = D^4 - 1$, rather than relying only on the trial-selection rule. Write the complete general solution.
4. Decide whether any solution is bounded on $[0, \infty)$. Prove your answer even when homogeneous constants are chosen to cancel all removable growing terms.

Question 4 – Solution

Strategy. Use linearity to separate polynomial forcing from two distinct resonant frequencies.

Step 1: Identify roots and trials. The characteristic roots are $1, -1, i, -i$, all simple, so

$$y_h = Ae^x + Be^{-x} + C \cos x + D \sin x.$$

Use $ax^2 + bx + c$ for $6x^2$, kxe^{-x} for $8e^{-x}$, and $x(m \cos x + n \sin x)$ for $10 \cos x$. Only the last two blocks are resonant.

Step 2: Compute the coefficients. The fourth derivative of a quadratic is zero, giving $a = -6, b = c = 0$. For a simple root λ of $P(r) = r^4 - 1$, the shift identity gives $P(D)(xe^{\lambda x}) = P'(\lambda)e^{\lambda x}$. Since $P'(-1) = -4$, we obtain $k = -2$.

For the trigonometric terms, $L(x \cos x) = 4 \sin x$ and $L(x \sin x) = -4 \cos x$. Therefore $m = 0, n = -5/2$, and

$$y_p = -6x^2 - 2xe^{-x} - \frac{5}{2}x \sin x.$$

Step 3: Verify all three residuals. Direct differentiation gives

$$L(-6x^2) = 6x^2, \quad D^4(xe^{-x}) = e^{-x}(x - 4),$$

$$D^4(x \sin x) = x \sin x - 4 \cos x.$$

Thus the second and third terms produce $8e^{-x}$ and $10 \cos x$ exactly. The full solution is $y = y_h + y_p$; no homogeneous constants are determined without additional data.

Step 4: Rule out bounded solutions. If $A \neq 0$, the Ae^x term dominates every other term, so boundedness is impossible. If $A = 0$, division by x^2 gives

$$\lim_{x \rightarrow \infty} \frac{y(x)}{x^2} = -6,$$

because the remaining oscillatory growth is only $O(x)$ and the other exponential terms decay. Hence even after removing the positive exponential, quadratic forcing remains. No solution is bounded on the positive half-line.