

Question 3

Consider the initially resting response of a fourth-order equation:

$$(D^2 + 1)^2 y = \cos x, \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Tasks

1. Identify the resonance multiplicity and choose a full real particular-solution trial. Explain why multiplying the ordinary sine-cosine trial by only x still fails.
2. Determine a particular solution by direct substitution, then add the complete homogeneous family.
3. Impose the four initial data and verify the resulting solution explicitly.
4. Evaluate the IVP solution at $x = 2\pi n$ and prove quadratic growth along this sequence. Explain why no homogeneous correction can make any solution bounded on $[0, \infty)$.

Question 3 – Solution

Strategy. Account for the double conjugate roots before imposing the initial state.

Step 1: Use the full resonance multiplicity. The roots $\pm i$ both have multiplicity two. The functions $\cos x, \sin x, x \cos x, x \sin x$ are all homogeneous, so an x -weighted trial still gives zero. Use $y_p = x^2(A \cos x + B \sin x)$.

Step 2: Match the forcing directly. A first application gives $(D^2 + 1)(x^2 \cos x) = 2 \cos x - 4x \sin x$, and a second gives $-8 \cos x$. Likewise, $(D^2 + 1)^2(x^2 \sin x) = -8 \sin x$. Thus $A = -1/8$, $B = 0$, and

$$y = -\frac{x^2 \cos x}{8} + (a + bx) \cos x + (c + dx) \sin x.$$

Step 3: Recover the initially resting solution. The particular term has initial vector $(0, 0, -1/4, 0)$. The homogeneous term has vector $(a, b + c, -a + 2d, -3b - c)$. Setting their sum to zero gives $a = b = c = 0, d = 1/8$, hence

$$y = \frac{x \sin x - x^2 \cos x}{8}.$$

The two numerator terms cancel to order two: $x \sin x - x^2 \cos x = x^4/3 + O(x^6)$ near zero. Thus y and its first three derivatives vanish there. The term $x \sin x/8$ is homogeneous, while the already checked particular term produces $\cos x$.

Step 4: Separate quadratic forcing from linear freedom. At $x = 2\pi n$,

$$y(2\pi n) = -\frac{(2\pi n)^2}{8}.$$

This proves unbounded quadratic growth in magnitude along that sequence. Every homogeneous term is $O(x)$ on the positive half-line, so no correction can cancel these quadratic values. The figure shows the response and the valid envelopes $\pm(x^2 + x)/8$.

