

Question 2

Consider the resonant third-order equation

$$(D - 1)^3 y = e^x(x^2 + 1)$$

on $\mathbb{R}$.

Tasks

1. Explain why $e^x(Ax^2 + Bx + C)$ cannot be a particular-solution trial. Give a complete resonance-corrected trial.
2. Use $y = e^x v$ to determine a particular solution by coefficient matching and write the full real general solution.
3. Solve the IVP $y(0) = y'(0) = y''(0) = 0$. Verify all three initial data and the original forcing.
4. For an arbitrary solution, calculate $\lim_{x \rightarrow \infty} e^{-x} y(x)/x^5$. Prove that no choice of homogeneous constants makes a solution bounded on $[0, \infty)$.

Question 2 – Solution

Strategy. Shift the triple root to zero, where the coefficient calculation becomes polynomial differentiation.

Step 1: Remove the homogeneous overlap. The characteristic root 1 has multiplicity three. Every proposed term $e^x(Ax^2 + Bx + C)$ is already homogeneous and gives zero under $(D-1)^3$. The corrected trial is

$$y_p = x^3 e^x (Ax^2 + Bx + C).$$

Step 2: Solve the shifted coefficient equations. Since $(D-1)(e^x v) = e^x v'$, the equation becomes $v''' = x^2 + 1$. For $v_p = Ax^5 + Bx^4 + Cx^3$, matching gives

$$60Ax^2 + 24Bx + 6C = x^2 + 1, \quad A = \frac{1}{60}, \quad B = 0, \quad C = \frac{1}{6}.$$

The complete solution is consequently

$$y = e^x \left(c_0 + c_1 x + c_2 x^2 + \frac{x^3}{6} + \frac{x^5}{60} \right).$$

Step 3: Impose and verify the initial data. With $v = e^{-x}y$, the initial values are

$$v(0) = y(0) = 0, \quad v'(0) = y'(0) - y(0) = 0, \quad v''(0) = y''(0) - 2y'(0) + y(0) = 0.$$

Thus $c_0 = c_1 = c_2 = 0$. The polynomial $v = x^3/6 + x^5/60$ and its first two derivatives vanish at zero, so the corresponding initial data for $y = e^x v$ vanish as well. Its third derivative is $v''' = 1 + x^2$, and the shift identity verifies the forcing exactly.

Step 4: Identify unavoidable resonant growth. For every choice of c_0, c_1, c_2 ,

$$\lim_{x \rightarrow \infty} \frac{e^{-x} y(x)}{x^5} = \frac{1}{60}.$$

The lower-degree terms disappear after division by x^5 . Hence $y(x)$ is eventually positive and asymptotic to $e^x x^5/60$, so it is unbounded. Homogeneous terms have polynomial degree at most two after removing e^x and cannot cancel the forced degree-five term.