

Question 1

Write $D = d/dx$ and let

$$L = D^2(D - 1)^3(D^2 + 4)^2.$$

Consider on $\mathbb{R}$ the equation

$$Ly = 3x^2 + (x + 1)e^x + x \cos 2x + e^{-x} \sin x.$$

This question concerns designing and justifying the coefficient trial, not carrying out a large elimination.

Tasks

1. State the order and complete real homogeneous solution, with all root multiplicities.
2. Write the standard undetermined-coefficient trial for a particular solution, including all polynomial degrees, both trigonometric partners, and every resonance factor. Count its unknown coefficients.
3. Explain why a trial containing only cosine terms for $x \cos 2x$, or only sine terms for $e^{-x} \sin x$, is not the appropriate full trial. Explain why one common resonance factor for all four forcing terms is inappropriate.
4. Prove that substitution determines exactly one member of your trial space. Use the effect of L on each polynomial-exponential family and its intersection with the homogeneous solution space. You need not compute the eleven coefficients.

Question 1 – Solution

Strategy. Treat each distinct exponential frequency separately and remove only its own homogeneous overlap.

Step 1: List the nine homogeneous modes. The roots are 0 twice, 1 three times, and $\pm 2i$ twice each. Thus L has order nine and

$$y_h = c_0 + c_1x + e^x(c_2 + c_3x + c_4x^2) + (c_5 + c_6x)\cos 2x + (c_7 + c_8x)\sin 2x.$$

Step 2: Build the trial block by block. A standard trial is

$$\begin{aligned} y_p = & x^2(a_2x^2 + a_1x + a_0) + x^3e^x(b_1x + b_0) \\ & + x^2((d_1x + d_0)\cos 2x + (e_1x + e_0)\sin 2x) \\ & + e^{-x}(f\cos x + g\sin x). \end{aligned}$$

There are $3 + 2 + 4 + 2 = \boxed{11}$ unknowns. The resonance powers are respectively 2, 3, 2, 0, because $-1 \pm i$ are not roots of L .

Step 3: Keep the derivative families complete. Differentiation mixes sine and cosine, and differentiating polynomial factors also generates lower powers of x . Therefore the full paired families must be allowed; coefficients that ultimately vanish should emerge from the equations. Resonance is tied to the particular exponent, not to the equation's total order. A common factor would either retain homogeneous overlap in some blocks or unnecessarily alter others.

Step 4: Establish solvability and uniqueness in the trial. For a root λ of multiplicity m , write the characteristic polynomial as $(r - \lambda)^m R(r)$ with $R(\lambda) \neq 0$. The shift identity gives

$$L(e^{\lambda x} x^m q) = e^{\lambda x} D^m R(D + \lambda)(x^m q).$$

For $\deg q \leq d$, the resulting polynomial has degree at most d ; its top coefficient is the top coefficient of q times $R(\lambda)(m + d)!/d!$, which is nonzero when that coefficient is nonzero. Successive coefficient matching is therefore invertible. Real and imaginary parts give the same conclusion for each paired block.

Equivalently, the eleven-dimensional trial maps into the eleven-dimensional forcing family and contains no nonzero homogeneous member: each block starts beyond its root's allowed homogeneous degree. Its kernel is zero, so this map is bijective. Hence the stated forcing determines exactly one trial member, although other particular solutions can differ by y_h .