

Question 10

For the fourth-order equation

$$y^{(4)} + 4y = 0,$$

define along a real solution

$$E(x) = y'''(x)y'(x) - \frac{1}{2}(y''(x))^2 + 2y(x)^2.$$

A quantity that is constant along solutions need not control their size.

Tasks

1. Find the characteristic roots and complete real general solution.
2. Differentiate E and prove that it is constant along every solution.
3. Verify that $y = e^x \cos x$ is an unbounded solution with $E = 0$. Give a specific sequence of times proving its unboundedness.
4. Characterize all solutions bounded on $[0, \infty)$ and all solutions bounded on $\mathbb{R}$. Explain why conservation of E does not contradict these classifications and cannot yield a bound of the form $|y(x)|^2 \leq CE(0)$ with a fixed positive C for all solutions.

Question 10 – Solution

Strategy. Compare the root-based behavior with what the conserved expression actually measures.

Step 1: Find growing and decaying oscillatory modes. Since $r^4 + 4 = (r^2 - 2r + 2)(r^2 + 2r + 2)$, the roots are $1 \pm i$ and $-1 \pm i$. Thus

$$y = e^x(A \cos x + B \sin x) + e^{-x}(C \cos x + D \sin x).$$

All roots are simple and the four real modes are independent.

Step 2: Verify the conservation law. The product rule gives exact cancellation:

$$E' = y^{(4)}y' + y'''y'' - y''y''' + 4yy' = y'(y^{(4)} + 4y) = 0.$$

Hence $E(x) = E(0)$ for every x .

Step 3: Test a zero-level growing solution. For $y = e^x \cos x$, the successive derivatives are

$$y' = e^x(\cos x - \sin x), \quad y'' = -2e^x \sin x, \quad y''' = -2e^x(\sin x + \cos x), \quad y^{(4)} = -4y.$$

At zero its initial vector is $(1, 1, 0, -2)$, so $E(0) = -2 + 2 = 0$. Conservation yields $E \equiv 0$. Yet $y(2\pi n) = e^{2\pi n} \rightarrow \infty$, proving unboundedness.

Step 4: Identify the limits of the invariant. On $[0, \infty)$ boundedness requires $A = B = 0$: any nonzero growing sinusoid attains its amplitude along an unbounded sequence, while the other term decays. These conditions also suffice, giving a two-dimensional decaying family. Requiring boundedness backward as well forces $C = D = 0$ by the same phase argument, so the only entire-line bounded solution is zero.

The expression E is not positive definite in the four initial derivatives: it has a mixed product and a negative square. In particular a nonzero state can have $E = 0$. The displayed solution has $|y(0)|^2 = 1$ and $E(0) = 0$, immediately refuting the proposed bound for every positive C . Constancy of an indefinite expression does not guarantee bounded motion.

