

Question 9

Multiplication by x is an operator: $(D - x)u = u' - xu$. On $\mathbb{R}$, consider

$$(D - x)D^2y = 0.$$

Unlike constant-coefficient factors, these factors need not commute.

Tasks

1. Expand the equation. Set $z = y''$ and find its complete real solution space, allowing definite integrals rather than elementary antiderivatives.
2. Define $F(x) = \int_0^x (x - t)e^{t^2/2} dt$. Prove that $1, x, F$ form a fundamental set by computing their Wronskian.
3. Solve the IVP $y(0) = y'(0) = 0, y''(0) = 1$. Prove that its solution is even and satisfies $y(x) \geq x^2/2$ for every real x , with strict inequality when $x \neq 0$.
4. Expand the reversed product $D^2(D - x)y$. Determine all functions that solve both the original and reversed equations, and explain why a characteristic polynomial cannot be used here as if x were constant.

Question 9 – Solution

Strategy. Solve the factors in their given order and use the product rule when reversing them.

Step 1: Reduce the original equation. Expansion gives $y''' - xy'' = 0$. Thus $z' - xz = 0$ and $z = Ce^{x^2/2}$. Integrating twice from zero yields

$$y = A + Bx + CF(x), \quad F(x) = \int_0^x (x-t)e^{t^2/2} dt.$$

This formula holds for negative x as well, with the usual oriented integral.

Step 2: Verify the fundamental set. Differentiating the integral gives $F' = \int_0^x e^{t^2/2} dt$ and $F'' = e^{x^2/2}$, so $F''' = xF''$. The derivative matrix is triangular:

$$W(1, x, F) = \det \begin{pmatrix} 1 & x & F \\ 0 & 1 & F' \\ 0 & 0 & e^{x^2/2} \end{pmatrix} = \boxed{e^{x^2/2} > 0}.$$

These are three independent solutions of a regular third-order linear equation, hence a fundamental set on $\mathbb{R}$.

Step 3: Fit and analyze the initial data. Since $F(0) = F'(0) = 0$ and $F''(0) = 1$, the IVP gives $\boxed{y = F}$. Its second derivative is even, and its value and slope at zero vanish; integrating shows that F is even. For $x > 0$,

$$F(x) - \frac{x^2}{2} = \int_0^x (x-t)(e^{t^2/2} - 1) dt > 0.$$

The inequality follows because the integrand is positive for $0 < t < x$. Evenness proves the same strict inequality for $x < 0$; equality holds at zero.

Step 4: Respect the order of the factors. The product rule gives

$$D^2(y' - xy) = y''' - xy'' - 2y'.$$

Subtracting the two equations forces $y' = 0$ for any common solution. Conversely, every constant solves both, so their common space consists exactly of constants. The variable coefficient creates the extra $-2y'$ term; treating x as a fixed characteristic root would discard it and give an invalid method.

