

Question 7

A real solution of

$$y^{(4)} + 5y'' + 4y = 0$$

is sampled at all times $x = n\pi$, $n = 0, 1, 2, \dots$. The measured values are 3 for even n and 1 for odd n .

Tasks

1. Find the complete real general solution and determine exactly what the value samples reveal about its four coefficients.
2. Describe the full subspace of solutions invisible to these samples, meaning $y(n\pi) = 0$ for every non-negative integer n . Prove that infinitely many value samples still leave nonuniqueness.
3. Two additional measurements give $y'(0) = 1$ and $y'(\pi) = 3$. Recover the unique solution and verify every type of measurement.
4. Prove that, for any solution of this equation, the four measurements $y(0), y(\pi), y'(0), y'(\pi)$ uniquely determine it. Explain why the equally spaced value samples alone miss actual oscillations between sampling times.

Question 7 – Solution

Strategy. Evaluate the real modes at the sampling grid before interpreting the amount of information in the data.

Step 1: Read the visible coefficients. The polynomial is $(r^2 + 1)(r^2 + 4)$, so

$$y = A \cos x + B \sin x + C \cos 2x + D \sin 2x.$$

At the sample times, $y(n\pi) = A(-1)^n + C$. Thus $A + C = 3$ and $-A + C = 1$, giving $A = 1$ and $C = 2$. These samples impose no restriction on B, D .

Step 2: Find the invisible subspace. Zero values at both even and odd sample indices force $A = C = 0$, while the sine modes vanish at every $n\pi$. Therefore

$$\{y : y(n\pi) = 0 \ \forall n \geq 0\} = \text{span}\{\sin x, \sin 2x\}.$$

It has dimension two. Adding any member to one fitting signal leaves every recorded value unchanged, even though the functions differ between samples.

Step 3: Recover the missing oscillations. The derivative samples give $B + 2D = 1$ and $-B + 2D = 3$, hence $B = -1, D = 1$. The unique signal is

$$y = \cos x - \sin x + 2 \cos 2x + \sin 2x.$$

Its value samples are $(-1)^n + 2$ and its derivative samples at $0, \pi$ are $1, 3$, respectively. Every term solves the equation.

Step 4: Prove general identifiability. For arbitrary measurements $u = y(0), v = y(\pi), p = y'(0)$ and $q = y'(\pi)$,

$$A = \frac{u - v}{2}, \quad C = \frac{u + v}{2}, \quad B = \frac{p - q}{2}, \quad D = \frac{p + q}{4}.$$

These formulas exist and are unique for every real measurement vector. Value samples alone repeat only two linear constraints; their number does not restore the sine information. The figure displays the two independent invisible modes and their common sample zeros.

