

Question 6

Let $\alpha \in \mathbb{R}$ and consider the third-order homogeneous equation

$$(D + 1)(D^2 + 2\alpha D + 1)y = 0$$

on $\mathbb{R}$. Call the equation *forward bounded* if every solution is bounded on $[0, \infty)$, and *forward decaying* if every solution tends to zero as $x \rightarrow +\infty$.

Tasks

1. Find the characteristic roots and identify all values of α where roots coincide, including coincidence with the fixed root -1 .
2. Write a complete real general solution in every parameter regime. Treat $\alpha = 1$ and $\alpha = -1$ explicitly.
3. Classify exactly the parameters for forward boundedness and forward decay. Prove the boundary cases rather than deciding from root real parts alone.
4. At $\alpha = 0, 1, -1$, respectively, give a nondecaying bounded solution, a decaying solution with a nonconstant polynomial factor, and an unbounded solution. Explain the different effects of repeated negative and repeated positive roots.

Question 6 – Solution

Strategy. Track the quadratic roots and separately check when either merges with the fixed root.

Step 1: Locate the collisions. Besides $r = -1$, the roots are $r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - 1}$. They coincide when $\alpha = \pm 1$. Substituting $r = -1$ in the quadratic gives $2 - 2\alpha$, so coincidence with the fixed root occurs only at $\alpha = 1$, producing a triple root -1 .

Step 2: State a complete basis in each case. For $|\alpha| > 1$, all three roots are real and distinct, and

$$y = Ce^{-x} + Ae^{r_+x} + Be^{r_-x}.$$

For $|\alpha| < 1$, let $\beta = \sqrt{1 - \alpha^2} > 0$. Then

$$y = Ce^{-x} + e^{-\alpha x}(A \cos \beta x + B \sin \beta x).$$

The exceptional cases are

$$\begin{aligned}\alpha = 1 : \quad y &= e^{-x}(C + Ax + Bx^2), \\ \alpha = -1 : \quad y &= Ce^{-x} + (A + Bx)e^x.\end{aligned}$$

Each family has three free constants.

Step 3: Classify all forward behavior. If $\alpha > 1$, both quadratic roots are negative since $\sqrt{\alpha^2 - 1} < \alpha$. If $0 < \alpha < 1$, their real part is $-\alpha < 0$. At $\alpha = 1$, even the quadratic polynomial factor is dominated by e^{-x} . Thus every solution decays for every $\alpha > 0$.

At $\alpha = 0$, the modes are $e^{-x}, \cos x, \sin x$: all are bounded, but not all decay. If $-1 < \alpha < 0$, the oscillatory amplitude grows exponentially; choosing a cosine and its phase maxima proves unboundedness. At $\alpha = -1$ there are positive-root modes, and for $\alpha < -1$ both quadratic roots are positive. Consequently,

forward bounded $\iff \alpha \geq 0$,	forward decaying $\iff \alpha > 0$.
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Step 4: Test the special parameters concretely. The requested examples are $\cos x$ at $\alpha = 0$, x^2e^{-x} at $\alpha = 1$, and xe^x at $\alpha = -1$. A finite polynomial factor does not defeat exponential decay at a negative root. A positive root already permits growth, and repetition adds further polynomial amplification.