

Question 5

For the fourth-order equation

$$y^{(4)} - 5y'' + 4y = 0,$$

write the initial derivatives as $(y(0), y'(0), y''(0), y'''(0)) = (a, b, c, d)$. For each simple characteristic root λ , define the polynomial

$$\Pi_\lambda(r) = \prod_{\mu \neq \lambda} \frac{r - \mu}{\lambda - \mu},$$

where the product runs over the other characteristic roots.

Tasks

1. Find the roots and prove that $\Pi_\lambda(D)y$ isolates the single mode $A_\lambda e^{\lambda x}$ in the general solution.
2. Compute the coefficients of e^x and e^{2x} explicitly from a, b, c, d using these polynomials.
3. Give necessary and sufficient conditions on the initial derivatives for boundedness on $[0, \infty)$. Express them also as two relations obtained from $(D+1)(D+2)y = 0$.
4. Start from $(a, b, c, d) = (1, -1, 1, -1)$ and perturb only d by a nonzero amount ε . Find the limit

$$\lim_{x \rightarrow \infty} e^{-2x} y_\varepsilon(x).$$

Explain the long-term consequence of arbitrarily small such errors.

Question 5 – Solution

Strategy. Use polynomial interpolation at the roots to extract growing modes directly from the initial data.

Step 1: Construct mode selectors. The polynomial factors as $(r-1)(r+1)(r-2)(r+2)$. Thus y is a linear combination of $e^{\lambda x}$ for $\lambda = -2, -1, 1, 2$. Since $Q(D)e^{\mu x} = Q(\mu)e^{\mu x}$ and $\Pi_\lambda(\mu)$ is 1 if $\mu = \lambda$ and 0 otherwise,

$$\Pi_\lambda(D)y = A_\lambda e^{\lambda x}.$$

Evaluating at zero reads off A_λ without solving all four coefficients simultaneously.

Step 2: Extract the unstable coefficients. The relevant polynomials are

$$\begin{aligned}\Pi_1(r) &= -\frac{(r+2)(r+1)(r-2)}{6} = \frac{-r^3 - r^2 + 4r + 4}{6}, \\ \Pi_2(r) &= \frac{(r+2)(r+1)(r-1)}{12} = \frac{r^3 + 2r^2 - r - 2}{12}.\end{aligned}$$

Hence

$$\boxed{A_1 = \frac{4a + 4b - c - d}{6}, \quad A_2 = \frac{d + 2c - b - 2a}{12}.$$

Step 3: Describe exactly the decaying initial states. Forward boundedness requires $A_2 = A_1 = 0$; distinct positive exponential rates cannot cancel. The remaining modes decay. Equivalently,

$$\boxed{c + 3b + 2a = 0, \quad d + 3c + 2b = 0.}$$

These follow from $y'' + 3y' + 2y = 0$ and its derivative for the two negative-root modes. Conversely, substituting these relations into A_1, A_2 makes both zero, so the relations are sufficient as well.

Step 4: Quantify the loss of cancellation. The unperturbed solution is e^{-x} . Replacing $d = -1$ by $-1 + \varepsilon$ produces $A_1 = -\varepsilon/6$ and $A_2 = \varepsilon/12$. All modes other than e^{2x} disappear after multiplication by e^{-2x} in the limit, giving

$$\boxed{\lim_{x \rightarrow \infty} e^{-2x} y_\varepsilon(x) = \varepsilon/12.}$$

Every nonzero error of this form eventually yields exponential growth in magnitude. Exact decay requires precise cancellation of both positive-root modes.