

Question 3

Consider

$$y^{(4)} + 8y'' + 16y = 0, \quad y(0) = y'(0) = y''(0) = 0, \quad y'''(0) = 8.$$

A repeated imaginary root differs substantially from a simple imaginary root.

Tasks

1. Find the characteristic roots and write the full real general solution, taking account of multiplicities.
2. Determine the unique IVP solution and verify its four initial data.
3. Evaluate the solution and its first derivative at $x_n = n\pi/2$, $n = 1, 2, \dots$. Use these values to prove unboundedness on $[0, \infty)$.
4. Characterize all solutions of the equation that are bounded on $[0, \infty)$. Prove that no nonzero solution tends to zero at infinity.

Question 3 – Solution

Strategy. Retain the polynomial factors attached to repeated conjugate roots and test growth at suitable phases.

Step 1: Build the real basis. The polynomial is $(r^2 + 4)^2$, with roots $\pm 2i$, each twice. Thus

$$y = (A + Bx) \cos 2x + (C + Dx) \sin 2x.$$

Omitting the factors $x \cos 2x$ and $x \sin 2x$ would lose half the solution space.

Step 2: Fit the initial vector. At zero the four derivatives are

$$A, \quad B + 2C, \quad -4A + 4D, \quad -12B - 8C.$$

The data give $A = D = 0$, $B = -1$ and $C = 1/2$, hence

$$y = \frac{1}{2} \sin 2x - x \cos 2x.$$

Indeed $y' = 2x \sin 2x$, $y'' = 2 \sin 2x + 4x \cos 2x$, and $y''' = 8 \cos 2x - 8x \sin 2x$, verifying all four data. Each basis term satisfies the factored equation.

Step 3: Exhibit growing extrema. For $x_n = n\pi/2$, we have

$$y'(x_n) = 0, \quad y(x_n) = (-1)^{n+1} \frac{n\pi}{2}, \quad y''(x_n) = 4x_n(-1)^n \neq 0.$$

These are alternating strict extrema with unbounded magnitude. In particular, purely imaginary roots alone do not imply boundedness when they are repeated.

Step 4: Classify the bounded family. If $(B, D) \neq (0, 0)$, choose a sequence $x \rightarrow \infty$ on which $B \cos 2x + D \sin 2x = \sqrt{B^2 + D^2} > 0$. The linear-in- x part grows, whereas $A \cos 2x + C \sin 2x$ is bounded. Therefore boundedness is equivalent to $B = D = 0$, a two-dimensional subspace. A nonzero member of that subspace attains its positive amplitude infinitely often, so it cannot tend to zero. An unbounded member cannot tend to zero either.

