

Question 10

Two solutions of $y^{(4)} = 0$ on $\mathbb{R}$ have initial derivative errors

$$\delta_j = \tilde{y}^{(j)}(0) - y^{(j)}(0), \quad |\delta_j| \leq \varepsilon \quad (j = 0, 1, 2, 3),$$

where $\varepsilon \geq 0$. Let $e = \tilde{y} - y$. We want a guaranteed error bound over a whole interval, not just at the initial point.

Tasks

1. Find $e(x)$ exactly in terms of the four initial errors.
2. For $T > 0$, prove the sharp uniform bound

$$\max_{|x| \leq T} |e(x)| \leq \varepsilon \left(1 + T + \frac{T^2}{2} + \frac{T^3}{6} \right).$$

Show that the constant cannot be improved when only the stated error information is available.

3. At $T = 2$, find the largest ε guaranteeing solution error at most 10^{-2} for every admissible initial error vector. Evaluate the guarantee for $\varepsilon = 10^{-3}$.
4. If only $\delta_0, \delta_1, \delta_2$ are controlled, can any finite uniform guarantee depending on those three errors alone hold on $[-2, 2]$? Construct a decisive example and interpret what the result says about a complete initial state.

Question 10 – Solution

Strategy. The linear initial-data map is explicit here, so its worst-case amplification can be determined exactly.

Step 1: Propagate all four initial errors. The difference satisfies $e^{(4)} = 0$ and $e^{(j)}(0) = \delta_j$. Therefore

$$e(x) = \delta_0 + \delta_1 x + \frac{\delta_2 x^2}{2} + \frac{\delta_3 x^3}{6}.$$

The factorials are required because the data are derivatives, not monomial coefficients.

Step 2: Prove and attain the uniform bound. For $|x| \leq T$, the triangle inequality gives

$$|e(x)| \leq \varepsilon \left(1 + |x| + \frac{|x|^2}{2} + \frac{|x|^3}{6} \right) \leq \varepsilon \left(1 + T + \frac{T^2}{2} + \frac{T^3}{6} \right).$$

Choose all four $\delta_j = \varepsilon$. At $x = T$, every term has the same nonnegative sign, attaining equality. Thus the stated worst-case constant is exact, including the trivial case $\varepsilon = 0$.

Step 3: Translate the certificate into a tolerance. At $T = 2$, the amplification factor is $1+2+2+8/6 = 19/3$. Consequently the largest admissible tolerance is

$$\varepsilon_{\max} = \frac{10^{-2}}{19/3} = \frac{3}{1900}.$$

Any larger value fails for the equal-sign error vector from Step 2. For $\varepsilon = 10^{-3}$, the exact worst-case bound is $19/3000 \approx 0.0063333 < 0.01$.

Step 4: Show why one uncontrolled component matters. Take $y = 0$ and $\tilde{y} = Mx^3/6$ for arbitrary real M . Both solve the equation and have the same value, first derivative and second derivative at zero, but their third-derivative difference is M . At $x = 2$, the solution error is $4|M|/3$, which is unbounded as $|M| \rightarrow \infty$. Thus controlling only three components cannot give the proposed finite guarantee. For the fourth-order IVP, the complete initial derivative vector is needed for this quantitative stability bound.