

Question 9

Consider the nonlinear third-order IVP

$$y''' = \sqrt{|y|}, \quad y(0) = y'(0) = y''(0) = 0.$$

For each delay $\tau \geq 0$, investigate a candidate that is zero for $x \leq \tau$ and has the form $C(x - \tau)^p$ for $x > \tau$, with $C > 0$ and $p > 3$.

Tasks

1. Explain why the standard linear IVP theorem does not apply, despite a nonzero leading coefficient and a continuous right-hand side.
2. Find the exponent p and coefficient C that make the positive branch satisfy the equation.
3. Verify the joined function as a classical solution on $\mathbb{R}$ and check all three initial data for every $\tau \geq 0$. Determine its exact smoothness at the joining point.
4. Explain how this family proves nonuniqueness. Contrast it with the linear equation $y''' = y$ under the same initial data, and examine the ratio $\sqrt{h}/h$ as $h \rightarrow 0^+$ to identify a regularity issue in the nonlinear dependence on y .

Question 9 – Solution

Strategy. Construct actual delayed solutions; continuity of a nonlinear right-hand side alone is not a uniqueness proof.

Step 1: Check linearity, not just the leading coefficient. The equation is third order but nonlinear because $\sqrt{|y|}$ is not a linear function of the unknown. A leading coefficient of 1 and continuity do not make the linear existence-and-uniqueness theorem applicable.

Step 2: Balance exponents and amplitudes. For $s = x - \tau > 0$, the proposed branch has $y''' = Cp(p-1)(p-2)s^{p-3}$ and $\sqrt{|y|} = \sqrt{C}s^{p/2}$. Since $p > 3$ and $C > 0$, equality requires $p-3 = p/2$, giving $p = 6$. Then $120C = \sqrt{C}$, so $C = 1/14400$.

Step 3: Verify the join and the initial state. Define

$$y_\tau(x) = \begin{cases} 0, & x \leq \tau, \\ (x - \tau)^6/14400, & x > \tau. \end{cases}$$

Derivatives through order five approach zero from the right and equal zero on the left; difference quotients at the join give the same derivatives. Thus $y_\tau \in C^5$, and the equation holds at the join too, since $y''' = \sqrt{|y|} = 0$ there. The sixth derivative jumps from 0 to $6!/14400 = 1/20$, so it is not C^6 at τ . Because $0 \leq \tau$, all three initial data vanish, including when $\tau = 0$.

Step 4: Identify the precise contrast. Distinct delays give distinct solutions with identical initial data; the identically zero function is another solution. This proves nonuniqueness constructively. For the linear equation $y''' - y = 0$, continuous normalized coefficients and the same zero initial vector force the unique zero solution. Here $\sqrt{h}/h = 1/\sqrt{h} \rightarrow \infty$, so the nonlinear right-hand side is not locally Lipschitz in y at zero. Nonlinearity alone does not always cause nonuniqueness; the displayed family establishes the failure in this specific case.

