

Question 8

For the homogeneous equation $y''' + y' = 0$ on $\mathbb{R}$, consider the three functions $1, \sin x, \cos x$. A student believes that a nonzero solution cannot have both its value and its first derivative zero at the same point.

Tasks

1. Verify the three functions as solutions and compute their Wronskian. Explain why they form a fundamental set on $\mathbb{R}$.
2. Find the solution with $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 1$, and verify the full initial state.
3. Determine all zeros of this solution and their exact multiplicities. Decide whether finitely or infinitely many such zeros are possible here.
4. Correct the student's claim for a regular homogeneous equation of general order n . State which consecutive derivatives must vanish at a point to force the entire solution to be zero, and explain why the present example does not satisfy that condition.

Question 8 – Solution

Strategy. Use the full initial derivative vector, rather than only the value and slope, to apply uniqueness.

Step 1: Verify a fundamental set. Each candidate satisfies $y''' + y' = 0$. Their Wronskian is

$$W = \det \begin{pmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{pmatrix} = \boxed{-1}.$$

Thus the three solutions are independent. The regular third-order equation has a three-dimensional homogeneous solution space, so they form a fundamental set.

Step 2: Determine the unique IVP solution. Write $y = A + B \sin x + C \cos x$. The data give $A + C = 0$, $B = 0$ and $-C = 1$. Hence

$$\boxed{y = 1 - \cos x.}$$

Its first two derivatives are $\sin x$ and $\cos x$, giving initial vector $(0, 0, 1)$; its third derivative $-\sin x$ verifies the equation.

Step 3: Classify all zero contacts. The equation $1 - \cos x = 0$ holds exactly at $x = 2k\pi$, $k \in \mathbb{Z}$. At each such point, $y' = 0$ but $y'' = 1 \neq 0$, so every zero has multiplicity exactly two. There are infinitely many, with no finite accumulation point. Their repeated tangency does not make the function identically zero.

Step 4: State the correct uniqueness consequence. For a normalized homogeneous linear equation of order n with continuous coefficients on a connected interval, if

$$\boxed{y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0,}$$

then uniqueness of the zero-data IVP forces $y \equiv 0$ throughout that interval. Here $n = 3$, and the second derivative at every zero is 1, so the required three-component vector is not zero. Value and slope alone are insufficient; an arbitrary number of separated double zeros is not excluded by the IVP theorem.

