

Question 7

Consider the singular third-order equation

$$xy''' - 2y'' = 0$$

on the whole real line. A classical solution here means a C^3 function satisfying the original equation at every point, including 0.

Tasks

1. Solve on each half-line by first setting $z = y''$. Display all constants before matching across zero.
2. Classify every global C^3 solution and determine the dimension of this solution space.
3. For prescribed $y(0) = \alpha$, $y'(0) = \beta$, $y''(0) = \gamma$, determine exactly when solutions exist and how many free parameters remain.
4. Determine which global solutions are C^4 . Does demanding this extra derivative restore uniqueness for compatible initial data at zero? Explain why the regular third-order dimension theorem cannot be used across this singular point.

Question 7 – Solution

Strategy. Solve separately where the leading coefficient is nonzero, then impose the actual regularity required at the join.

Step 1: Solve on both regular components. For $x \neq 0$, $xz' - 2z = 0$ gives $z = Cx^2$. Two integrations, with the constant absorbed into the quartic coefficient, yield

$$y = \begin{cases} A_+x^4 + B_+x + C_+, & x > 0, \\ A_-x^4 + B_-x + C_-, & x < 0. \end{cases}$$

The constants on the two components are initially independent.

Step 2: Impose a classical join. Continuity of y and y' forces $C_+ = C_- = C$ and $B_+ = B_- = B$. The second and third derivatives are $12A_{\pm}x^2$ and $24A_{\pm}x$, both tending to zero, so C^3 regularity imposes no relation between A_+ and A_- . Every global solution is therefore

$$y(x) = C + Bx + \begin{cases} A_+x^4, & x \geq 0, \\ A_-x^4, & x < 0, \end{cases}$$

with four independent constants. At zero the equation reduces to $-2y''(0) = 0$, which these functions satisfy. This is a four-dimensional solution space.

Step 3: Classify the initial data. Every solution has $y(0) = C$, $y'(0) = B$ and $y''(0) = 0$. Thus $\boxed{\gamma = 0}$ is necessary and sufficient. When it holds, $C = \alpha$, $B = \beta$ and both A_+, A_- remain arbitrary. When $\gamma \neq 0$, no classical solution exists.

Step 4: Check the extra derivative. The one-sided fourth derivatives are $24A_+$ and $24A_-$. A C^4 join requires and is guaranteed by $A_+ = A_- = A$, giving the polynomial $C + Bx + Ax^4$. One free parameter A still survives for the compatible initial data, so uniqueness is not restored. The leading coefficient vanishes at 0, and normalization introduces $-2/x$. The hypotheses guaranteeing dimension three and arbitrary unique initial data hold only on the separate half-lines.