

Question 6

For $y^{(4)} = 0$ on $[0, 1]$, prescribe four scalar endpoint conditions:

$$y(0) = 0, \quad y(1) = 0, \quad y'(0) + y'(1) = 0, \quad y''(1) - y''(0) = \eta,$$

where η is a real parameter. Counting four conditions is not the same as verifying that they determine four constants independently.

Tasks

1. Write the general solution and translate all four conditions into a linear system for its coefficients. Determine the rank of the coefficient matrix.
2. Classify all η according to existence and uniqueness, and give every solution whenever solutions exist.
3. Replace the fourth condition by $y'(0) = 1$, keeping the first three. Find the resulting solution and prove it is unique.
4. Explain why none of these conclusions contradicts the standard fourth-order IVP theorem. Describe how the family in the compatible original problem varies while retaining all four endpoint conditions.

Question 6 – Solution

Strategy. Test the endpoint functionals on the cubic solution space instead of relying on their number.

Step 1: Form the coefficient system. Write $y = a + bx + cx^2 + dx^3$. The conditions become

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta \end{pmatrix}.$$

The fourth row is six times the third minus twelve times the second plus twelve times the first. The first three rows are independent, so the rank is $\boxed{3}$.

Step 2: Classify the parameter completely. The first three equations imply $a = 0$, $b + c + d = 0$ and $2b + 2c + 3d = 0$; subtracting twice the middle equation gives $d = 0$. Thus the last equation can hold only if $\eta = 0$. For that value, $c = -b$ and

$$y = Kx(1 - x), \quad K \in \mathbb{R}.$$

There are infinitely many solutions if $\eta = 0$ and none if $\eta \neq 0$. No value of η gives a unique solution under the original conditions.

Step 3: Replace the redundant information. The condition $y'(0) = 1$ gives $b = 1$. The first three conditions already imply $a = d = 0$ and $c = -b$, so

$$y = x(1 - x)$$

is the unique solution after replacement. The new condition fixes the one remaining free parameter.

Step 4: Distinguish endpoint data from a full initial state. The IVP theorem concerns y, y', y'', y''' specified at one ordinary point. These endpoint conditions mix values at two points and can be dependent or inconsistent on the solution space. In the compatible family, changing K changes height and curvature while preserving zero endpoint values, opposite endpoint slopes, and equal endpoint curvatures. The figure shows three such members; their different initial derivative vectors explain why IVP uniqueness is not violated.

