

Question 4

Four functions $y_1, \dots, y_4$ solve the same homogeneous equation

$$y^{(4)} - \frac{2x}{1+x^2}y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0$$

on $\mathbb{R}$, where p_0, p_1, p_2 are continuous. Their Wronskian at zero is $W(0) = -6$.

Tasks

1. Derive the Wronskian identity $W' = -p_3W$ for a normalized fourth-order homogeneous equation, by differentiating the determinant row by row.
2. Determine $W(x)$ for the stated equation and decide whether the four functions form a fundamental set on the whole real line.
3. Explain why every four-component initial derivative vector at any finite x_0 is realized by exactly one linear combination of these functions.
4. A computation reports $\widetilde{W}(x) = -6(1+x^2)(1-x)$. It has the correct value at 0. Give two independent reasons why it cannot be the Wronskian of this common solution family.

Question 4 – Solution

Strategy. The coefficient of the next-to-highest derivative controls the entire Wronskian, not just its value at one point.

Step 1: Differentiate the determinant. The Wronskian rows contain derivative orders 0, 1, 2, 3. Differentiating any of the first three rows duplicates the following row, giving determinant zero. In the last-row term, substitute $y_j^{(4)} = -p_3 y_j''' - p_2 y_j'' - p_1 y_j' - p_0 y_j$ for every column. All terms except the multiple of the original last row again duplicate a row. Therefore

$$W' = -p_3 W.$$

Step 2: Determine the Wronskian globally. Here $p_3 = -2x/(1+x^2)$, so

$$W(x) = -6 \exp\left(\int_0^x \frac{2s}{1+s^2} ds\right) = -6(1+x^2).$$

It never vanishes. The four solutions are linearly independent, and the regular fourth-order homogeneous solution space has dimension four; hence they form a fundamental set on $\mathbb{R}$.

Step 3: Interpret invertibility of the derivative matrix. At any finite x_0 , the matrix with columns $(y_j, y_j', y_j'', y_j''')(x_0)$ has determinant $W(x_0) \neq 0$. It therefore maps one and only one coefficient vector to any prescribed initial derivative vector. The resulting combination solves the equation; the linear IVP theorem shows it is the unique solution for those data on $\mathbb{R}$.

Step 4: Reject a plausible-looking candidate. First, $\widetilde{W}(1) = 0$ contradicts the nonvanishing formula forced by $W(0) \neq 0$ on this regular connected interval. Second, it fails the differential identity even at the initial point: $\widetilde{W}'(0) = 6$, whereas $-p_3(0)\widetilde{W}(0) = 0$. Matching one determinant value is insufficient; a valid Wronskian must obey the identity throughout the interval.