

Question 3

For $y^{(4)} = 0$ on $\mathbb{R}$, consider the shifted functions

$$f_0 = 1, \quad f_1 = x - 1, \quad f_2 = \frac{(x-1)^2}{2}, \quad f_3 = \frac{(x-1)^3}{6}.$$

Initial derivatives and polynomial coefficients coincide only for an appropriately normalized basis.

Tasks

1. Verify that the four functions solve the equation and compute their Wronskian for arbitrary x .
2. Construct the solution with $y(1) = 2$, $y'(1) = -1$, $y''(1) = 3$ and $y'''(1) = 4$. Explain why the coefficients in this basis can be read directly from the data.
3. Express the same solution in the unshifted basis $1, x, x^2, x^3$ and verify the four data in that form.
4. Prove that the map sending a solution to $(y(1), y'(1), y''(1), y'''(1))$ is a linear isomorphism onto $\mathbb{R}^4$. What does this say about the number of free constants and the effect of changing the basis?

Question 3 – Solution

Strategy. Build the derivative matrix and use its normalization at the initial point.

Step 1: Verify solutions and a nonzero determinant. Every function is cubic or lower, so its fourth derivative is zero. With $h = x - 1$, their derivative matrix is

$$F(x) = \begin{pmatrix} 1 & h & h^2/2 & h^3/6 \\ 0 & 1 & h & h^2/2 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is triangular with unit diagonal, giving $\boxed{W = \det F = 1}$ at every x .

Step 2: Use the identity matrix at the center. Since $F(1) = I$, the coefficient vector is exactly the prescribed derivative vector. Therefore

$$\boxed{y = 2 - h + \frac{3}{2}h^2 + \frac{2}{3}h^3, \quad h = x - 1.}$$

The factorials in the basis make its initial derivative matrix the identity; without them, derivative values would not equal coefficients.

Step 3: Change coordinates without changing the solution. Expansion gives

$$\boxed{y = \frac{23}{6} - 2x - \frac{1}{2}x^2 + \frac{2}{3}x^3.}$$

Here $y' = -2 - x + 2x^2$, $y'' = -1 + 4x$, $y''' = 4$. At 1, these are $-1, 3, 4$, while $y(1) = 2$, verifying all data independently of the shifted expression.

Step 4: Establish the initial-data isomorphism. Differentiation and evaluation are linear. For any vector in $\mathbb{R}^4$, the corresponding combination of f_0, f_1, f_2, f_3 realizes it, proving surjectivity. If all four derivatives at 1 vanish, the coefficient vector in this basis is zero, proving injectivity. Thus the solution space has dimension four. A change of basis changes coordinates, not the number of free constants, the initial data themselves or the resulting function.