

Question 2

Let $L[y] = y^{(4)}$ and consider $L[y] = 24x$ on $\mathbb{R}$. Define

$$p(x) = \frac{x^5}{5}, \quad u(x) = p(x) + 1 + x^2, \quad v(x) = p(x) + 2x - x^3.$$

For a nonhomogeneous equation, the phrase “superposition of solutions” needs a precise qualification.

Tasks

1. Verify that p, u, v solve the equation. Determine whether $u + v$, $u - v$ and $(u + v)/2$ solve the same equation or its homogeneous counterpart.
2. Find every triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ for which $\alpha u + \beta v + \gamma p$ solves $L[y] = 24x$.
3. Prove that the complete solution set is p plus the space of cubic polynomials. Explain why it is an affine set but not a vector space.
4. Find the unique solution with $y(0) = 1$, $y'(0) = -2$, $y''(0) = 6$, $y'''(0) = 0$, and verify the forcing and all four data.

Question 2 – Solution

Strategy. Apply the linear operator before deciding which combinations preserve the nonzero forcing.

Step 1: Track the right-hand side. Since $p^{(4)} = 24x$ and fourth derivatives of cubics vanish, $L[p] = L[u] = L[v] = 24x$. Linearity gives

$$L[u + v] = 48x, \quad L[u - v] = 0, \quad L[(u + v)/2] = 24x.$$

Thus the sum solves neither of the two stated equations, the difference solves the homogeneous equation, and the average solves the original one.

Step 2: Characterize every admissible combination. The proposed combination has image $24x(\alpha + \beta + \gamma)$. Because $24x$ is not identically zero on $\mathbb{R}$,

$$\boxed{\alpha u + \beta v + \gamma p \text{ solves the equation} \iff \alpha + \beta + \gamma = 1.}$$

The forcing's isolated zero at 0 does not remove this identity requirement.

Step 3: Describe the complete solution set. If y is any solution, $(y - p)^{(4)} = 0$. Four integrations show that $y - p$ is a polynomial of degree at most three. Conversely, adding any such polynomial to p preserves the equation. Hence

$$\boxed{y = p + c_0 + c_1x + c_2x^2 + c_3x^3.}$$

This is a translate of a four-dimensional vector space, hence affine. It is not itself a vector space: the zero function does not satisfy the nonzero forcing, and sums generally double the forcing.

Step 4: Fit and check the initial state. The data give $c_0 = 1$, $c_1 = -2$, $2c_2 = 6$, $6c_3 = 0$, so

$$\boxed{y = \frac{x^5}{5} + 1 - 2x + 3x^2.}$$

The added cubic has fourth derivative zero. Its value and first three derivatives at 0 are $1, -2, 6, 0$; p contributes zero to all four. The coefficients are uniquely fixed.