

Question 1

Consider the real differential equation

$$(x^2 - 1)y^{(4)} + 2xy''' - 3y' = \ln x.$$

A leading coefficient that vanishes at one point requires more care than simply counting derivatives.

Tasks

1. Classify the equation by order, linearity and homogeneity. Write its normalized form wherever that form is defined over the reals.
2. At $x_0 = 1/2$, prescribe $y = 1$, $y' = 0$, $y'' = -2$ and $y''' = 3$. State the largest interval containing x_0 on which the standard linear IVP theorem guarantees a unique solution, and explain why four data are needed.
3. A proposed C^4 solution near $x = 1$ has $y(1) = 0$, $y'(1) = 2$, $y''(1) = 0$, $y'''(1) = 1$. Test these data directly in the original equation.
4. Replace only the last datum by $y'''(1) = 3$. Explain what this compatibility check establishes and what it does not establish. Does a vanishing leading coefficient make the original equation third order on a neighborhood of 1?

Strategy. Separate the equation's order from the intervals on which its highest derivative can be isolated.

$$y^{(4)} + \frac{2x}{x^2 - 1}y''' - \frac{3}{x^2 - 1}y' = \frac{\ln x}{x^2 - 1}.$$

Step 2: Apply the theorem on its proper interval. All normalized coefficients and the forcing are continuous on $\boxed{(0,1)}$, the largest such interval through $1/2$. The linear IVP theorem gives one solution for arbitrary values of y, y', y'', y''' at that ordinary point. These four data specify the fourth-order initial state. The theorem does not assert that this particular solution cannot extend beyond the interval; it simply gives no guarantee across 1.

$$2y'''(1) - 3y'(1) = 0.$$

Step 4: Interpret compatibility without overclaiming. With $y'''(1) = 3$, the same test gives $6 - 6 = 0$. This removes that contradiction, but alone proves neither existence nor uniqueness; the normalized IVP theorem still does not apply at 1. The coefficient of $y^{(4)}$ is nonzero at nearby points other than 1, so the equation remains fourth order on a neighborhood. At the isolated point, its value imposes a lower-derivative constraint instead of determining $y^{(4)}(1)$.

