

Question 10

For $x \geq 1$, consider a finite interval of forcing in an Euler equation:

$$x^2 y'' - xy' + y = g(x), \quad y(1) = y'(1) = 0, \quad g(x) = \begin{cases} x, & 1 \leq x < e, \\ 0, & x \geq e. \end{cases}$$

Seek a solution that is C^1 across $x = e$ and solves the equation on each side. No classical second derivative at the switching point is assumed.

Tasks

1. Set $t = \ln x$ and $Y = e^t Z$. Derive a general definite-integral solution for zero initial data, valid for piecewise continuous forcing g .
2. Apply it to the specified forcing and give explicit formulas on $1 \leq x \leq e$ and $x \geq e$.
3. Verify continuity of y and y' at e . Compute the one-sided second derivatives and explain why a C^2 solution at that point is impossible.
4. Determine the large- x behavior after the forcing stops. Explain why the solution does not return to zero just because the right-hand side becomes zero.

Question 10 – Solution

Strategy. Remove the repeated exponent, integrate in logarithmic time, and match the finite-forcing transition.

Step 1: Derive the integral response. With $D = d/dt$, the transformed equation is $(D-1)^2 Y = g(e^t)$, so $Z'' = e^{-t}g(e^t)$. The data at $t = 0$ give $Z(0) = Z'(0) = 0$. Thus

$$Z(t) = \int_0^t (t-s)e^{-s}g(e^s)ds, \quad \boxed{y(x) = x \int_1^x \ln(x/r) \frac{g(r)}{r^2} dr.}$$

The second formula follows from $r = e^s$, including $ds = dr/r$.

Step 2: Evaluate the finite forcing interval. Here $Z'' = 1$ for $0 < t < 1$ and $Z'' = 0$ for $t > 1$. Integrating with the zero initial data gives $Z = t^2/2$ up to 1, then $Z = t - 1/2$. Therefore

$$\boxed{y(x) = \begin{cases} \frac{1}{2}x(\ln x)^2, & 1 \leq x \leq e, \\ x(\ln x - \frac{1}{2}), & x \geq e. \end{cases}}$$

Both formulas have the same value at e , and the first satisfies both initial conditions at 1.

Step 3: Check the join and the derivative jump. For $Y = e^t Z$, $y' = Z + Z'$ and $y'' = (Z' + Z'')/x$. At $t = 1$, both sides have $Z = 1/2$ and $Z' = 1$, so

$$y(e) = e/2, \quad y'(e) = 3/2, \quad y''(e^-) = 2/e, \quad y''(e^+) = 1/e.$$

The finite jump in forcing produces a jump $-1/e$ in y'' , while y, y' remain continuous. The unequal one-sided limits rule out C^2 regularity, irrespective of any value assigned to $g(e)$.

Step 4: Interpret the surviving homogeneous motion. After e , the solution is the nonzero homogeneous combination $x \ln x - x/2$. Thus $\boxed{y(x)/(x \ln x) \rightarrow 1}$ as $x \rightarrow \infty$. The completed forcing interval leaves nonzero value and slope at the switch. Those become initial data for the homogeneous equation; zero subsequent forcing does not reset them to zero.