

Question 9

For real α and $\beta > 0$, consider a nonzero solution on $x > 0$ of

$$x^2 y'' + (1 - 2\alpha)xy' + (\alpha^2 + \beta^2)y = 0.$$

Its power envelope may tend to zero even while its derivatives keep oscillating.

Tasks

1. Find the real general solution and show that its j th derivative has the form $x^{\alpha-j}$ times a sinusoid in $\ln x$ with nonzero amplitude.
2. For each integer $k \geq 0$, prove a necessary and sufficient condition on α for a nonzero solution to have a C^k extension to $x = 0$. Treat the equality case explicitly.
3. Apply your criterion to $y = x^{5/2} \sin(\ln x)$. Compute y'' and determine exactly which integer orders of smoothness are possible at zero.
4. Join this particular solution to zero for $x \leq 0$. Verify that the joined function is a classical C^2 solution of the corresponding Euler equation on the whole real line. Explain the implication for initial data at the singular point.

Question 9 – Solution

Strategy. Track both the power envelope and the nonvanishing oscillation amplitude after every differentiation.

Step 1: Differentiate the complex powers. The exponents are $\alpha \pm i\beta$, so the real solutions are

$$y = x^\alpha (A \cos(\beta \ln x) + B \sin(\beta \ln x)).$$

The j th derivative of $x^{\alpha+i\beta}$ is $\prod_{\ell=0}^{j-1} (\alpha - \ell + i\beta) x^{\alpha-j+i\beta}$. Since $\beta > 0$, none of these factors vanishes. A nonzero real solution therefore retains a nonzero sinusoidal amplitude at every derivative order.

Step 2: Establish the exact regularity threshold. If $\alpha > k$, all derivatives through order k tend to zero. Their extensions are consistent derivatives: for $j < k$, $y^{(j)}(x)/x = O(x^{\alpha-j-1}) \rightarrow 0$. If $\alpha = k$, the k th derivative has constant nonzero amplitude and no limit. If $\alpha < k$, that derivative has an unbounded envelope and attains unbounded values along a sequence approaching zero. Hence

$$\boxed{\text{A nonzero solution extends as } C^k \text{ to } 0 \iff \alpha > k.}$$

This includes $k = 0$; bounded oscillation at equality is not continuity.

Step 3: Inspect the concrete example. For $\alpha = 5/2$, $\beta = 1$,

$$\boxed{y'' = x^{1/2} \left(\frac{11}{4} \sin(\ln x) + 4 \cos(\ln x) \right).}$$

The solution extends as C^0, C^1, C^2 , with all three center values zero, but not as C^3 or any higher integer order. The plot shows $y''(e^t)$ and its decaying envelope as $t \rightarrow -\infty$.

Step 4: Verify the joined classical solution. The equation is $x^2 y'' - 4xy' + (29/4)y = 0$. On $x > 0$ the formula solves it; on $x < 0$ the zero function does. The zero limits through the second derivative make the join C^2 , and at 0 every term in the original equation is zero. Both this nonzero joined solution and the identically zero solution have $y(0) = y'(0) = 0$. Classical IVP uniqueness fails at this singular point.

