

Question 8

For fixed $a \in \mathbb{R}$ and parameter $\varepsilon \geq 0$, consider on $x > 0$

$$x^2 y'' + (1 - 2a)xy' + (a^2 - \varepsilon^2)y = 0.$$

Two distinct power solutions become nearly identical when their characteristic exponents approach one another.

Tasks

1. Find the characteristic exponents for $\varepsilon > 0$ and explain why simply taking their two power solutions to the limit does not produce a fundamental pair at $\varepsilon = 0$.
2. Construct solutions $u_\varepsilon, v_\varepsilon$ satisfying $u_\varepsilon(1) = 1$, $u'_\varepsilon(1) = a$, $v_\varepsilon(1) = 0$, $v'_\varepsilon(1) = 1$.
3. Compute their Wronskian and prove independence for every $\varepsilon \geq 0$, using the limiting pair at zero.
4. Determine that limiting pair and justify uniform convergence of each basis function on every compact interval $[r, R] \subset (0, \infty)$ as $\varepsilon \rightarrow 0$. Explain how the logarithmic repeated-root solution emerges.

Question 8 – Solution

Strategy. Normalize a symmetric sum and a divided difference before allowing the characteristic roots to merge.

Step 1: Identify the degenerating pair. The power equation is $(m - a)^2 - \varepsilon^2 = 0$, giving $m = a \pm \varepsilon$. Both $x^{a+\varepsilon}$ and $x^{a-\varepsilon}$ tend to x^a as $\varepsilon \rightarrow 0$. Their separate limits are the same function, so they cannot remain a fundamental pair.

Step 2: Normalize before taking the limit. For $\varepsilon > 0$, set $t = \ln x$ and define

$$u_\varepsilon = x^a \cosh(\varepsilon t), \quad v_\varepsilon = x^a \frac{\sinh(\varepsilon t)}{\varepsilon}.$$

These are the half-sum and the difference divided by 2ε of the two powers. At $x = 1$ they have the four required value/slope data by direct differentiation.

Step 3: Check the determinant exactly. Differentiation and $\cosh^2 s - \sinh^2 s = 1$ give

$$W(u_\varepsilon, v_\varepsilon) = x^{2a-1} > 0.$$

At $\varepsilon = 0$, define $u_0 = x^a$ and $v_0 = x^a \ln x$. They solve the repeated-root equation, have the same normalized data, and their Wronskian is again x^{2a-1} . Independence holds for every parameter value.

Step 4: Justify the limit uniformly. On $[r, R]$, let $T = \max(|\ln r|, |\ln R|)$ and $M = \max_{[r, R]} x^a$. For $0 < \varepsilon \leq 1$, Taylor's theorem for $\cosh$ and $\sinh$ gives

$$|u_\varepsilon - u_0| \leq \frac{M\varepsilon^2 T^2 \cosh T}{2}, \quad |v_\varepsilon - v_0| \leq \frac{M\varepsilon^2 T^3 \cosh T}{6}.$$

Both bounds tend to zero uniformly. The divided difference tends to $\partial_m x^m|_{m=a} = x^a \ln x$, recovering the independent logarithmic solution that the two unnormalized limits lose.