

Question 7

Let $L > 0$ be fixed. Seek nonzero real solutions of the two-endpoint Euler problem

$$x^2 y'' + xy' + \lambda y = 0, \quad y(1) = y(e^L) = 0,$$

where λ is real. No general theory of boundary-value eigenproblems is required; solve the Euler equation directly.

Tasks

1. Transform the problem to $0 \leq t \leq L$ and prove that $\lambda \leq 0$ permits only the zero solution.
2. Find every $\lambda > 0$ permitting a nonzero solution, and give the corresponding solution families.
3. For each admissible value, impose $y'(1) = 1$. Find the normalized solution and all its zeros, including the number strictly inside $(1, e^L)$.
4. Explain why uniqueness of ordinary initial-value problems does not imply uniqueness for these two endpoint conditions. For $L = \pi$, describe the first three normalized profiles in the coordinate $t = \ln x$.

Question 7 – Solution

Strategy. In logarithmic coordinates, the two endpoint conditions reduce to elementary trigonometric or hyperbolic equations.

Step 1: Exclude nonpositive parameters. Set $Y(t) = y(e^t)$. Then $Y'' + \lambda Y = 0$, with $Y(0) = Y(L) = 0$. If $\lambda = 0$, $Y = A + Bt$ and both constants must vanish. If $\lambda = -\mu^2 < 0$, $Y = A \cosh(\mu t) + B \sinh(\mu t)$. The first condition gives $A = 0$; since $\sinh(\mu L) > 0$, the second gives $B = 0$.

Step 2: Find all nonzero families. For $\lambda = \omega^2 > 0$, the first condition leaves $Y = B \sin(\omega t)$. A nonzero B requires $\sin(\omega L) = 0$, so

$$\lambda_n = (n\pi/L)^2, \quad y = C \sin\left(\frac{n\pi \ln x}{L}\right), \quad n = 1, 2, \dots$$

There are no other real parameter values admitting nonzero solutions.

Step 3: Normalize and locate the zeros. At $x = 1$, $y'(1) = Y'(0) = Cn\pi/L$. Thus

$$y_n(x) = \frac{L}{n\pi} \sin\left(\frac{n\pi \ln x}{L}\right).$$

Its zeros on the specified interval are $x_k = e^{kL/n}$ for $k = 0, \dots, n$. Exactly $n - 1$ lie strictly inside; they are equally spaced in $\ln x$, not in x .

Step 4: Distinguish endpoint and initial data. The original endpoint conditions do not specify the initial slope. At an admissible parameter, every amplitude satisfies both endpoints, including zero; hence those data are not unique. Once $y'(1) = 1$ is imposed, the usual IVP uniqueness applies. For $L = \pi$, the first three normalized profiles are $Y_n(t) = \sin(nt)/n$, $n = 1, 2, 3$, with 0, 1, 2 interior zeros, respectively. The figure uses logarithmic horizontal coordinates.

