

Question 6

For $x > 0$, consider

$$x^2 y'' + xy' = \frac{x^2}{1+x^2}.$$

Instead of prescribing data at a positive point, seek the solution that is bounded near 0 and extends continuously with $y(0) = 0$.

Tasks

1. Integrate the equation to obtain its general solution in definite-integral form on $(0, \infty)$, displaying both arbitrary constants.
2. Impose boundedness and the specified value at zero. Prove that they select exactly one solution on the positive half-line.
3. Derive its power series about zero, give its first three nonzero terms, and determine the radius and both endpoint outcomes. Justify termwise integration.
4. Prove that the selected solution has a C^2 extension to zero, compute $y'(0)$ and $y''(0)$, and certify a two-term approximation on $[0, 1]$ using a rigorous remainder bound.

Question 6 – Solution

Strategy. Integrate the weighted derivative first, then use regularity at zero to remove the logarithmic homogeneous mode.

Step 1: Integrate the Euler operator. Dividing by x gives $(xy')' = x/(1+x^2)$. Hence

$$xy' = \frac{1}{2} \ln(1+x^2) + C, \quad y = \frac{1}{2} \int_1^x \frac{\ln(1+s^2)}{s} ds + C \ln x + D.$$

These formulas display both constants and solve the equation for every $x > 0$.

Step 2: Apply the singular-endpoint conditions. Since $\ln(1+s^2)/s \sim s$ at zero, its integral there converges. Boundedness therefore forces $C = 0$; the value $y(0) = 0$ fixes the remaining constant. The unique selected solution is

$$y(x) = \frac{1}{2} \int_0^x \frac{\ln(1+s^2)}{s} ds.$$

Any two bounded solutions differ by a constant, and their common value at zero makes that constant zero.

Step 3: Expand the integral with justification. The divided logarithmic series $\sum_{n \geq 1} (-1)^{n-1} s^{2n-1}/n$ is uniformly convergent on $|s| \leq \rho < 1$, by comparison with $\sum_{n \geq 1} \rho^{2n-1}/n$. Termwise integration therefore yields, for $|x| < 1$,

$$y(x) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n^2} = \frac{x^2}{4} - \frac{x^4}{16} + \frac{x^6}{36} - \dots$$

The ratio of successive nonzero terms tends to x^2 , so the radius is 1. At $x = \pm 1$ the series converges absolutely by comparison with $\sum 1/n^2$. The series and the even integral extension agree there by continuity: the series is uniformly convergent on $[-1, 1]$ by the same comparison.

Step 4: Recover derivatives and a certificate. The local power series supplies a smooth even extension with $y'(0) = 0$ and $y''(0) = 1/2$. With the forcing defined continuously as zero at 0, the original equation holds there as well. On $[0, 1]$, the alternating term magnitudes decrease, so for $P_4 = x^2/4 - x^4/16$,

$$0 \leq y(x) - P_4(x) \leq \frac{x^6}{36}.$$

The bound includes the endpoint $x = 1$; it is a certificate, not a claim that the two-term approximation is highly accurate there.