

Question 5

For $x > 0$, solve the resonantly forced Euler problem

$$x^2 y'' - 3xy' + 4y = x^2 \ln x, \quad y(1) = y'(1) = 0.$$

A trial expression $x^2(A + B \ln x)$ resembles the forcing but may not produce it.

Tasks

1. Transform the equation using $t = \ln x$ and then $Y = e^{2t}Z$. Derive the resulting equation for Z .
2. Find the general solution. Explain exactly why the proposed trial expression fails and why a cubic logarithmic term appears.
3. Apply the initial conditions and verify the final solution by computing the transformed residual.
4. Determine whether the IVP solution extends continuously, as a C^1 function, or as a C^2 function to 0. Give the relevant limiting expressions.

Question 5 – Solution

Strategy. Remove the repeated characteristic exponential before integrating the forcing.

Step 1: Factor the logarithmic-time operator. The equation is $(D - 2)^2 Y = e^{2t}t$, where $D = d/dt$. Since $(D - 2)(e^{2t}Z) = e^{2t}Z'$, a second application gives

$$Z'' = t.$$

Step 2: Integrate through the resonance. Two integrations yield $Z = t^3/6 + C_1 + C_2t$, hence

$$y = x^2 \left(C_1 + C_2 \ln x + \frac{(\ln x)^3}{6} \right).$$

The proposed trial has $Z = A + Bt$ and therefore $Z'' = 0$: it belongs entirely to the homogeneous solution space. Integrating the degree-one forcing twice produces degree three, accounting for the cubic logarithm and its factor $1/6$.

Step 3: Impose and verify the initial data. At $x = 1$, $t = 0$, so $y(1) = C_1$ and $y'(1) = 2C_1 + C_2$. Thus both constants vanish and

$$y = \frac{x^2(\ln x)^3}{6}.$$

For this solution, $Z = t^3/6$ satisfies $Z'' = t$ exactly; therefore the original left-hand side is $e^{2t}t = x^2 \ln x$. The two initial values are zero.

Step 4: Inspect regularity at the singular point. Writing $t = \ln x$,

$$y' = x \left(\frac{t^3}{3} + \frac{t^2}{2} \right), \quad y'' = \frac{t^3}{3} + \frac{3t^2}{2} + t.$$

As $x \rightarrow 0^+$, the power of x dominates every logarithmic power, so $y \rightarrow 0$, $y' \rightarrow 0$, and $y(x)/x \rightarrow 0$. Assigning $y(0) = 0$ therefore gives a C^1 right extension, which may be joined to zero on the left as a C^1 function. But $y'' \rightarrow -\infty$, so no C^2 extension is possible.