

Question 4

Solve on the negative half-line:

$$x^2 y'' + xy' - \frac{1}{4}y = 0, \quad y(-4) = 3, \quad y'(-4) = -\frac{1}{8}.$$

Using fractional powers without specifying a real branch can invalidate a solution.

Tasks

1. Use $t = \ln(-x)$ to derive a real fundamental pair on $x < 0$. Explain why the positive-half-line expression $x^{1/2}$ is unsuitable as a real-valued basis there.
2. Translate both initial values into logarithmic coordinates and solve the IVP. Check the derivative sign directly in the original variable.
3. Give its maximal real interval and both endpoint behaviors.
4. Find its unique stationary point and minimum. Explain how increasing logarithmic time relates to increasing x on this negative interval.

Question 4 – Solution

Strategy. Work with positive $-x$, retaining the sign in the chain rule.

Step 1: Choose a real branch. For $x = -e^t$, $dx/dt = x$, so $Y' = xy'$ and $Y'' = xy' + x^2y''$ still hold. The equation becomes $Y'' - Y/4 = 0$. A real pair is $\boxed{(-x)^{1/2}, (-x)^{-1/2}}$ for $x < 0$. The expression $x^{1/2}$ has no real value on this interval; powers of $|x| = -x$ avoid that problem.

Step 2: Translate the slope correctly. At $t = \ln 4$, $Y = 3$ and $Y' = xy' = (-4)(-1/8) = 1/2$. Writing $y = A\sqrt{-x} + B/\sqrt{-x}$ yields $2A + B/2 = 3$ and $A - B/4 = 1/2$, hence $A = 1$, $B = 2$. Thus

$$\boxed{y = \sqrt{-x} + \frac{2}{\sqrt{-x}}, \quad y' = -\frac{1}{2\sqrt{-x}} + \frac{1}{(-x)^{3/2}}.}$$

At -4 these give 3 and $-1/8$, as required.

Step 3: State the maximal interval. The solution is smooth throughout $(-\infty, 0)$, and its divergence at 0 prevents extension through that point. This is the maximal real interval containing -4 . As $x \rightarrow -\infty$, $y \sim \sqrt{-x} \rightarrow \infty$; as $x \rightarrow 0^-$, $y \sim 2/\sqrt{-x} \rightarrow \infty$.

Step 4: Locate the minimum and orient time. Put $r = -x > 0$. The derivative is $(2 - r)/(2r^{3/2})$, so it is negative for $x < -2$ and positive for $-2 < x < 0$. Therefore

$$\boxed{x_* = -2, \quad y(x_*) = 2\sqrt{2}}$$

is the unique global minimum. Since $dt/dx = 1/x < 0$, increasing x corresponds to decreasing t on this half-line. This reverses the relation between the two horizontal directions, not the derivative formulas.

