

Question 1

Consider the Euler initial-value problem

$$x^2y'' - 2xy' - 4y = 0, \quad y(1) = 2, \quad y'(1) = 3.$$

A power x^m is a useful trial function, but the reason it works should also be understood.

Tasks

1. For $x > 0$, set $t = \ln x$ and $Y(t) = y(e^t)$. Derive the formulas for xy' and x^2y'' and transform the equation into one with constant coefficients.
2. Find a fundamental pair, compute its Wronskian, and solve the IVP exactly. Verify both initial values.
3. State the maximal real interval of this IVP and determine the behavior of its solution at each end of that interval.
4. Locate and classify every stationary point on that interval. Give the exact minimum value and explain how two monotone power modes can combine into a nonmonotone solution.

Question 1 – Solution

Strategy. Logarithmic coordinates turn powers into exponentials and make the characteristic equation systematic.

Step 1: Transform derivatives correctly. The chain rule gives $Y' = xy'$ and $Y'' = xy' + x^2y''$, hence

$$xy' = Y', \quad x^2y'' = Y'' - Y'.$$

The equation becomes $Y'' - 3Y' - 4Y = 0$, with characteristic polynomial $m^2 - 3m - 4 = (m - 4)(m + 1)$.

Step 2: Solve and verify the data. The powers x^4, x^{-1} form a pair with $W = x^4(-x^{-2}) - 4x^3x^{-1} = -5x^2 \neq 0$ for $x > 0$. Writing $y = C_1x^4 + C_2x^{-1}$ gives $C_1 + C_2 = 2$, $4C_1 - C_2 = 3$, so

$$y = x^4 + x^{-1}.$$

At 1, its value is 2 and derivative $4x^3 - x^{-2}$ is 3.

Step 3: Identify the maximal interval. The normalized coefficients are continuous on $(0, \infty)$, and the explicit solution exists throughout it. Since $y \rightarrow +\infty$ as $x \rightarrow 0^+$, no finite continuation through 0 is possible. Thus the maximal real interval through 1 is $(0, \infty)$. At infinity, $y \sim x^4 \rightarrow +\infty$; at zero, $y \sim x^{-1}$.

Step 4: Find the unique minimum. The equation $y' = 0$ is $4x^5 = 1$, giving $x_* = 4^{-1/5}$. Since $y'' = 12x^2 + 2x^{-3} > 0$, it is the unique global minimum. Using $x_*^{-1} = 4x_*^4$,

$$y(x_*) = 5 \cdot 4^{-4/5}.$$

One mode increases and the other decreases. Their derivatives cancel at exactly one point, although neither mode has a stationary point of its own.

