

Question 8

An unknown equation has the form

$$y'' + \alpha xy' + (\beta + \gamma x^2)y = 0,$$

where α, β, γ are real constants. Two analytic solutions have measured Taylor expansions

$$u(x) = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + O(x^8), \quad v(x) = x - x^3 + \frac{1}{2}x^5 - \frac{1}{6}x^7 + O(x^9).$$

The displayed coefficients are exact.

Tasks

1. Determine α, β, γ by substituting only the coefficients needed from these data. Show the recovery is unique.
2. Derive the full recurrence for a generic series solution of the recovered equation, including its conventions at the lowest indices.
3. Use the substitution $y = e^{-x^2}w$ to identify every solution exactly. Verify that the two measured series are a fundamental pair and determine their radii of convergence.
4. Predict the coefficients of x^8 in u and x^9 in v . Can an additional exact measurement assigning the value $1/20$ to the former coefficient be consistent with the equation and the original data?

Question 8 – Solution

Strategy. Infer the differential operator from low-order data, then validate all orders by a transformation.

Step 1: Recover the three constants. The constant coefficient in the equation for u gives $-2 + \beta = 0$. The x coefficient for v gives $-6 + \alpha + \beta = 0$. The x^2 coefficient for u gives $6 - 2\alpha - \beta + \gamma = 0$. These successive equations uniquely force

$$\boxed{\beta = 2, \quad \alpha = 4, \quad \gamma = 4.}$$

Step 2: State a recurrence valid from the start. With $a_{-2} = a_{-1} = 0$ for the shifted contribution,

$$\boxed{a_{n+2} = -\frac{(4n+2)a_n + 4a_{n-2}}{(n+2)(n+1)} \quad (n \geq 0).}$$

The starting coefficients $a_0 = y(0)$ and $a_1 = y'(0)$ are free. In particular, $a_2 = -a_0$ and $a_3 = -a_1$; no negative-index coefficient is an additional parameter.

Step 3: Verify the recovered equation globally. Writing $E = e^{-x^2}$, differentiation gives

$$y' = E(w' - 2xw), \quad y'' = E(w'' - 4xw' + (4x^2 - 2)w).$$

Thus $y'' + 4xy' + (2 + 4x^2)y = Ew''$. The equation becomes $w'' = 0$, yielding

$$\boxed{y = e^{-x^2}(C_1 + C_2x), \quad u = e^{-x^2}, \quad v = xe^{-x^2}.}$$

Their initial data match the measured series, and their Wronskian is $E^2 = e^{-2x^2} > 0$. Both functions are entire, so both Taylor radii are infinite. The transformation also proves the entire measured series, not just the first displayed terms, are consistent.

Step 4: Test a new exact measurement. The exponential expansion gives $[x^8]u = 1/4! = 1/24$ and $[x^9]v = 1/24$. The recurrence produces the same values. Since the recovered parameters and each solution's initial data are fixed uniquely, $1/20$ is incompatible; it cannot be accommodated by choosing later coefficients independently.