

Question 7

Consider the nonlinear initial-value problem

$$y' = 1 + y^2, \quad y(0) = 0.$$

Seek a Maclaurin solution $y = \sum_{n \geq 0} a_n x^n$. For the radius question, you may use these facts: sine and cosine are entire; the only complex zeros of cosine are $z = \pi/2 + k\pi$, $k \in \mathbb{Z}$; and a Taylor series reaches its nearest nonremovable complex singularity.

Tasks

1. Derive the nonlinear coefficient recurrence using a Cauchy product. Compute all coefficients through degree seven.
2. Prove that the resulting solution is odd, and explain why the square y^2 cannot be handled by squaring individual coefficients separately.
3. Identify the exact solution by a method independent of the coefficient calculation. Determine its maximal real interval containing 0 and the exact Taylor radius.
4. For the degree-seven polynomial P_7 , prove $0 < P_7(x) < y(x)$ when $0 < x < \pi/2$. Explain why a fixed polynomial cannot approximate this solution uniformly on that whole open interval.

Question 7 – Solution

Strategy. Nonlinearity couples coefficients by convolution; an exact solution then validates convergence and reveals blow-up.

Step 1: Convolve rather than square termwise. The constant forcing contributes only at $n = 0$:

$$(n+1)a_{n+1} = \delta_{n0} + \sum_{k=0}^n a_k a_{n-k}, \quad a_0 = 0,$$

where $\delta_{n0} = 1$ for $n = 0$ and 0 otherwise. This gives

$$P_7(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315},$$

with $a_2 = a_4 = a_6 = 0$.

Step 2: Establish parity and coefficient signs. If y solves the IVP, $-y(-x)$ does too; local uniqueness gives oddness. The recurrence also proves by induction that all even coefficients vanish and all odd coefficients are positive: each required convolution contains positive odd-index products. For example, the coefficient of x^4 in y^2 is $2a_1a_3$, not a_2^2 . Cross terms are essential.

Step 3: Identify the function and its radius. Separation yields $\arctan y = x$ near 0, so $y = \tan x$. It satisfies the equation and initial condition on $(-\pi/2, \pi/2)$ and blows up at both endpoints, making this the maximal real interval through 0. Since $\sin(\pm\pi/2) \neq 0$, the nearest zeros of cosine are nonremovable poles of the quotient. The supplied facts give $R = \pi/2$. Within this radius, the analytic solution's coefficients obey the recurrence, which uniquely determines them; hence it is the constructed series. At $x = \pi/2$ the positive-term series diverges: a finite endpoint sum would bound $\tan x$ below that endpoint, contradicting blow-up. Oddness gives divergence at $-\pi/2$ as well.

Step 4: Interpret truncation and blow-up. For $0 < x < \pi/2$, every omitted odd term is positive, so $0 < P_7(x) < \tan x$. But P_7 stays bounded as $x \rightarrow (\pi/2)^-$ while $\tan x \rightarrow +\infty$. Their difference is unbounded on that interval. Every fixed Taylor truncation has this defect; convergence on smaller compact intervals does not imply uniform approximation up to a blow-up boundary.

