

Question 6

The leading coefficient vanishes at the proposed expansion center in

$$xy'' + 2y' + xy = 0.$$

A student therefore needs to check which initial data a Maclaurin solution can actually accommodate.

Tasks

1. Substitute $y = \sum_{n \geq 0} a_n x^n$. Find the exceptional constant-coefficient equation and the recurrence for all remaining coefficients.
2. Determine every analytic solution near 0 and identify its series in terms of a familiar function, including the value assigned at 0.
3. Decide whether $y(0) = 1$, $y'(0) = 1$ can be satisfied by an analytic solution or even by a twice continuously differentiable solution of the original equation near 0.
4. On $x > 0$, substitute $v = xy$ to find the full two-parameter solution. Determine which members extend boundedly to 0 and explain why only one free parameter survives in the analytic family.

Question 6 – Solution

Strategy. Apply the original equation at its singular center before invoking any ordinary-point intuition.

Step 1: Retain the exceptional equation. The coefficient of x^0 is $2a_1 = 0$. For $n \geq 1$, the coefficient of x^n is $(n+1)(n+2)a_{n+1} + a_{n-1} = 0$. Equivalently,

$$a_1 = 0, \quad a_{m+2} = -\frac{a_m}{(m+2)(m+3)} \quad (m \geq 0).$$

Thus a_0 is free but a_1 is not.

Step 2: Identify all analytic solutions. Every odd coefficient vanishes. The even chain gives

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!} = a_0 \frac{\sin x}{x} \quad (x \neq 0), \quad y(0) = a_0.$$

The ratio test gives infinite radius. Termwise substitution verifies the original equation, including $x = 0$. Conversely, every analytic solution must obey the recurrence, so the displayed family is exhaustive.

Step 3: Test the proposed initial values. The data $y(0) = 1$, $y'(0) = 1$ are incompatible with $a_1 = 0$. More generally, a C^2 solution satisfies the equation at 0, where it reduces to $2y'(0) = 0$. Even if the equation were initially imposed only at nearby nonzero points, continuity of y, y', y'' would give the same constraint by taking the limit. Such a C^2 solution cannot have the proposed data.

Step 4: Explain the lost second parameter. With $v = xy$, $v'' = xy'' + 2y'$, so $v'' + v = 0$ for $x > 0$. Consequently

$$y(x) = A \frac{\sin x}{x} + B \frac{\cos x}{x} \quad (x > 0).$$

The second term behaves like B/x as $x \rightarrow 0^+$; boundedness forces $B = 0$. The remaining term extends analytically with value A and slope 0. The ordinary-point theorem does not guarantee two arbitrary analytic initial coefficients at a center where the leading coefficient is zero.