

Question 5

For real λ , consider

$$(1 - x^2)y'' - 2xy' + \lambda y = 0$$

and seek analytic solutions near $x = 0$. A polynomial solution would remain meaningful even where the leading coefficient vanishes.

Tasks

1. Derive the recurrence and explain how it separates even and odd coefficients.
2. Classify all real λ for which a nonzero polynomial solution exists. Prove necessity from the highest-degree term and sufficiency from the recurrence; include degree zero.
3. For $\lambda = 20$, $y(0) = 3/8$ and $y'(0) = 0$, find the exact solution and verify the differential equation by substitution.
4. Find all its real zeros exactly, determine its values at $x = \pm 1$, and explain why this particular solution extends through those points although the normalized equation is singular there.

Question 5 – Solution

Strategy. Use coefficient termination to distinguish polynomial solutions from general local series.

Step 1: Derive the parity recurrence. Matching x^n gives

$$a_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)} a_n \quad (n \geq 0).$$

Even and odd indices evolve independently from a_0 and a_1 .

Step 2: Characterize termination completely. If a nonzero polynomial has degree m , its highest coefficient obeys $[\lambda - m(m+1)]a_m = 0$, so $\lambda = m(m+1)$ for an integer $m \geq 0$. Conversely, choose the starting coefficient of parity m nonzero and the other zero. For indices of that parity below m , no numerator vanishes; at $n = m$ it does, terminating the chain. Thus these and only these parameter values work, including $m = 0$, $\lambda = 0$ (constants).

Step 3: Construct and verify the requested polynomial. For $\lambda = 20$, the recurrence yields $a_2 = -15/4$, $a_4 = 35/8$, and $a_6 = 0$. Hence

$$y(x) = \frac{35x^4 - 30x^2 + 3}{8}.$$

Here $y' = (35x^3 - 15x)/2$ and $y'' = (105x^2 - 15)/2$. Substitution in $(1 - x^2)y'' - 2xy' + 20y$ cancels every coefficient. Initial data also hold, so uniqueness near 0 identifies the solution.

Step 4: Locate roots and extend the solution. Solving the quadratic in x^2 gives four distinct zeros:

$$x = \pm \sqrt{\frac{15 - 2\sqrt{30}}{35}}, \quad x = \pm \sqrt{\frac{15 + 2\sqrt{30}}{35}}.$$

They lie in $(-1, 1)$, and $y(-1) = y(1) = 1$. The polynomial satisfies the original equation identically for every real x , including ± 1 . Singular normalized coefficients do not forbid this special solution from extending through them.

