

Question 4

Consider a nonhomogeneous equation centered away from the origin:

$$y'' + (x - 2)y' = e^{x-2}, \quad y(2) = 0, \quad y'(2) = 1.$$

You may use the analytic ordinary-point theorem: a linear equation with nonzero leading coefficient and analytic normalized coefficients and forcing in a complex disk has a convergent Taylor solution throughout that disk.

Tasks

1. Set $h = x - 2$ and derive the recurrence for $y = \sum_{n \geq 0} a_n h^n$, explicitly accounting for the constant coefficient of hy' .
2. Compute a_0 through a_6 and the resulting polynomial $P_6(h)$. Identify any coefficient that vanishes despite nonzero neighboring coefficients.
3. Define the residual $\mathcal{R} = P_6'' + hP_6' - e^h$. Find its leading nonzero term. Explain why a degree-six solution truncation need not satisfy the differential equation through degree six.
4. Produce a definite-integral representation of the exact solution and justify infinite radius of convergence for its series. Check both initial values in your representation.

Question 4 – Solution

Strategy. Expand every term in the shifted variable and keep the forcing coefficients in the same index convention.

Step 1: Match the forcing coefficient. Derivatives with respect to x and h agree. The coefficient of h^n in hy' is na_n for $n \geq 0$; it is zero when $n = 0$. Thus

$$a_{n+2} = \frac{1/n! - na_n}{(n+2)(n+1)} \quad (n \geq 0), \quad a_0 = 0, \quad a_1 = 1.$$

Step 2: Compute the initial segment. Successive applications give $a_2 = 1/2$, $a_3 = 0$, $a_4 = -1/24$, $a_5 = 1/120$, $a_6 = 1/144$. Therefore

$$P_6(h) = h + \frac{h^2}{2} - \frac{h^4}{24} + \frac{h^5}{120} + \frac{h^6}{144}.$$

The cubic coefficient vanishes because $1/1! - a_1 = 0$, not because the solution has a fixed parity.

Step 3: Interpret the residual correctly. Direct differentiation cancels the coefficients through h^4 . At degree five, hP'_6 contributes $1/24$ while e^h contributes $1/120$; hence

$$\mathcal{R}(h) = \frac{h^5}{30} + O(h^6).$$

The omitted term a_7h^7 would contribute $42a_7h^5$ to y'' . Indeed, the recurrence gives $a_7 = -1/1260$, cancelling that residual term. Two differentiations lower the degree of an omitted term by two.

Step 4: Recover and validate the exact solution. Writing $v = y'$ gives $v' + hv = e^h$. An integrating factor yields

$$y(2+h) = \int_0^h e^{-s^2/2} \left(1 + \int_0^s e^{t+t^2/2} dt \right) ds.$$

At $h = 0$ the outer integral is zero and its integrand is 1, verifying the initial values. Differentiating gives the first-order equation for v , hence the original equation. Its leading coefficient is 1 and all coefficients and forcing are entire, so the stated theorem applies to disks of every radius. The Taylor series is therefore entire.